Universidade de Aveiro Departamento de Matemática
2014

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Numerical and Combinatorial applications of generalized Appell polynomials

Aplicações numéricas e combinatórias de polinómios de Appell generalizados

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Tese apresentada à Universidade de Aveiro para cumprimento dos requisitos necessários à obtenção do grau de Doutor em Matemática, realizada sob a orientação científica do Doutor Helmuth Malonek, Professor Catedrático do Departamento de Matemática da Universidade de Aveiro e co-orientação da Doutora Maria Irene Ferrão Carvalho Ribeiro Almeida Falcão, Professora Associada do Departamento de Matemática e Aplicações da Universidade do Minho.

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palavras-chave

Funções holomorfas hipercomplexas, polinómios de Appell generalizados, polinómios homogéneos holomorfos, transformações de Joukowski generalizadas, transformações quase-conformes, n-simplex de Pascal com entradas hipercomplexas, variáveis totalmente regulares, potências generalizadas, potências pseudo-complexas, matrizes de Vandermonde, identidades combinatórias.

resumo

Esta tese estuda propriedades e aplicações de diferentes polinómios de Appell generalizados no contexto da análise de Clifford.

Exemplificando uma transformação realizada por polinómios de Appell generalizados, é introduzida uma transformação análoga à transformação de Joukowski complexa de ordem dois. A análise de um n- simplex de Pascal com entradas hipercomplexas permite sublinhar a relevância combinatória de polinómios hipercomplexos de Appell.

O conceito de variáveis totalmente regulares e a sua relação com polinómios de Appell generalizados conduz à construção de novas bases para o espaço dos polinómios homogéneos holomorfos cujos elementos são todos isomorfos às potências inteiras da variável complexa. Por este motivo, tais polinómios são chamados de potências pseudo-complexas (PCP). Diferentes variantes de PCP são objeto de uma investigação detalhada.

É dada especial atenção aos aspectos numéricos de PCP. Um algoritmo eficiente baseado em aritmética complexa é proposto para a sua implementação. Neste contexto, é apresentado um breve resumo de métodos numéricos para inverter matrizes de Vandermonde e é proposto um algoritmo modificado para ilustrar as vantagens de um tipo especial de PCP.

Finalmente, são enfatizadas aplicações combinatórias de polinómios de Appell generalizados. A expressão explícita dos coeficientes de um tipo particular de polinómios de Appell e a sua relação com um simplex de Pascal com entradas hipercomplexas são obtidas. A comparação de dois tipos de polinómios de Appell tridimensionais leva à deteção de novas fórmulas envolvendo somas trigonométricas e de identidades combinatórias do tipo de Riordan – Sofo, caracterizadas pela sua expressão em termos de coeficientes binomiais centrais.

keywords

Clifford holomorphic functions, generalized Appell polynomials, homogeneous holomorphic polynomials, generalized Joukowski transformation, quasi-conformal mappings, Pascal n-simplex with hypercomplex entries, totally regular variables, generalized powers, pseudo-complex powers, Vandermonde matrices, combinatorial identities.

abstract

This thesis studies properties and applications of different generalized Appell polynomials in the framework of Clifford analysis.

As an example of 3D-quasi-conformal mappings realized by generalized Appell polynomials, an analogue of the complex Joukowski transformation of order two is introduced. The consideration of a Pascal n-simplex with hypercomplex entries allows stressing the combinatorial relevance of hypercomplex Appell polynomials.

The concept of totally regular variables and its relation to generalized Appell polynomials leads to the construction of new bases for the space of homogeneous holomorphic polynomials whose elements are all isomorphic to the integer powers of the complex variable. For this reason, such polynomials are called pseudo-complex powers (PCP). Different variants of them are subject of a detailed investigation.

Special attention is paid to the numerical aspects of PCP. An efficient algorithm based on complex arithmetic is proposed for their implementation. In this context a brief survey on numerical methods for inverting Vandermonde matrices is presented and a modified algorithm is proposed which illustrates advantages of a special type of PCP.

Finally, combinatorial applications of generalized Appell polynomials are emphasized. The explicit expression of the coefficients of a particular type of Appell polynomials and their relation to a Pascal simplex with hypercomplex entries are derived. The comparison of two types of 3D Appell polynomials leads to the detection of new trigonometric summation formulas and combinatorial identities of Riordan-Sofo type characterized by their expression in terms of central binomial coefficients.

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Introduction

[...] At first, the thing seemed to me to be based more on sophism than on truth, but I searched until I found the proof.

R. Bombelli

Besides of its long history, classical complex function theory (CFT) is until today one of the most central areas in mathematics. It is a rich and beauty theory in its own right that contributed with fundamental concepts to the development of other modern mathematical subjects. Moreover, CFT is also remarkably useful in a wide range of applied mathematics. Its generalization to a theory of several complex variables (cf. [93]), naturally connected with functions of an even number of real variables greater than 2, saw remarkable advances since the 30-ties of the last century and is now one of the cornerstones of algebraic geometry.

However, the 30-ties saw also the birth of an alternative approach to the generalization of CFT in the form of a *hypercomplex function theory* (HFT)[1]. Particularly important are the facts that HFT is not necessarily connected with functions of an even number of real variables and relies, in general, on the use of noncommutative algebras. Its first systematic steps can be traced back to the work of R. Fueter[2] on quaternion valued functions of a quaternion variable. At the end of his live Fueter studied also Clifford holomorphic functions defined in $\mathbb{R}^{n+1}$ with values in a noncommutative Clifford Algebra (cf.[66]). Instead of using the designation *Clifford holomorphic* functions for functions in the kernel of a suitably generalized Cauchy-Riemann or generalized Dirac operators, the name *monogenic* is also widely used (cf. [76]). Meanwhile results and methods developed in the framework of HFT have been applied

[1] A hypercomplex number is a traditional term for an element of an algebra over the field of real numbers. In this sense, hypercomplex analysis is the extension of real analysis and complex analysis to the study of functions where the argument is a hypercomplex number. The use of the term *hypercomplex function theory* follows Rudolf Fueter (1870-1950) and some recent trend in the field of *Clifford Analysis* dedicated to its roots in CFT-problems.

[2] More about the historical background of HFT and their protagonists can be found in Section 1.5.

to a wide variety of other areas, such as PDE's, quasi-conformal mappings, combinatorics, mathematical physics, signal processing, robotics, computer vision, neural computing, among others.

The motivation for carrying out research on different types and applications of Clifford algebra-valued generalized Appell polynomials was influenced by two main facts.

First of all we would like to mention its actuality. The concept of generalized Clifford holomorphic Appell polynomials has been introduced for the first time in the framework of Clifford Analysis less than ten years ago (cf. [43, 55]). Their construction was the solution to the long lasting open problem of search for Clifford holomorphic polynomials that behave with respect to the hypercomplex derivative exactly in the same way as complex power functions with respect to the complex derivative. In some sense this was also an answer to the general question of the existence of a suitable *homogeneous Clifford holomorphic function* of degree k, that in diverse contexts could play the role of the (unfortunately) *non-monogenic* k-th power ($k = 1, 2, \ldots$), of the ordinary hypercomplex variable

$$x = x_0 + x_1 e_1 + \cdots + x_n e_n,$$

where $\{e_1, e_2, \ldots, e_n\}$ is an orthonormal basis of $\mathbb{R}^n$ ($n \geq 2$), subject to the multiplication rules

$$\begin{aligned} e_i e_j &= -e_j e_i, \quad i \neq j, \quad i, j = 1, \ldots, n, \\ e_i^2 &= -1. \end{aligned}$$

Soon after their introduction generalized Clifford holomorphic Appell polynomials have been studied by several other authors and from different points of view (e.g. [7, 17, 18, 82, 118]). This concerns also the construction of orthogonal polynomial Appell bases for generalized Fourier series expansions of monogenic functions. Worth noting that the general representation theoretic treatment, realized e.g. in [18, 21] by relying on the concept of Gelfand -Tsetlin bases, confirmed the exceptional role of the special sequence of generalized monogenic Appell polynomials presented in [43, 55]. The results presented in Chapter 2 are obtained as application of this special Appell sequence to 3D quasi-conformal mappings.

The second fact concerns the systematical use of a construction principle of a set of generalized monogenic Appell polynomials, not considered so far, and the study of its impact on numerical and combinatorial applications. In the center of our attention was a type of Appell polynomials of arbitrary homogenous degree k isomorphic to z^k, $z \in \mathbb{C}$, $k = 1, 2, \ldots$ and therefore designated as *pseudo-complex powers* (PCP). Besides studying their properties for the case of $n+1$ real variables, we paid particular attention to the 3D case ($n=2$). In concrete, we studied the numerical costs involved in the construction of these polynomials and proved their numerical efficiency when compared with the well known generalized powers (in [76] also called *Fueter polynomials*).

As another example of applications of PCP we derived summation formulas of trigonometrical functions inherent in the parameterized form of PCP as well as binomial identities as results of bijective representations. We show that the use of the underlying noncommutative algebra can lead to generalizations of known results but, at the same time, opens the way for deriving new useful formulas and combinatorial identities.

This thesis is organized in six chapters.

Chapter 1 contains the preliminaries on the Clifford Algebra $\mathcal{C}\ell_{0,n}$ over the real Euclidean space $\mathbb{R}^n$, identifying the complex algebra $\mathbb{C}$ with $\mathcal{C}\ell_{0,1}$. We consider two different types of hypercomplex structures of $\mathbb{R}^{n+1}$, both presented in detail in [101]. Each of them may be interpreted as higher dimensional *complexification* of the real Euclidean space $\mathbb{R}^{n+1}$ similar to the well known association of $\mathbb{R}^2$ with $\mathbb{C}$. The concepts of Clifford holomorphic function and its hypercomplex derivative are introduced and some examples of Clifford holomorphic and non-holomorphic functions are discussed. The Taylor series expansion of a Clifford holomorphic function in terms of the so-called *Generalized Powers* (GP) is then presented. A corresponding version of the Cauchy-Kowalewskaya extension is also referred. In view of their applications, some Clifford Algebra software packages are mentioned, with particular emphasis to those which are available for the algebra of real quaternions. The last section provides some brief historical remarks on Clifford algebras as well as on hypercomplex function theory.

Chapter 2 is dedicated to the study of generalized Appell sequences of Clifford holomorphic polynomials. Following [43, 55] the consideration of these sequences is motivated by the fact that positive integer powers of the usually considered hypercomplex variable are not holomorphic, except for the complex case. In this context, we first recall a special type of Appell polynomials introduced by Falcão and Malonek in [43], here designated by *Standard Appell Polynomials* (SAP). Different representations of lower degree polynomials are referred, as well as their coefficients properties stressing some combinatorial relations. Thereafter, we extend Appell sequences to holomorphic power functions of negative degree, by using the Kelvin transform of the SAP. As an application we generalize the results of Almeida and Malonek [46] about the Joukowski transformation in higher dimensions by considering *generalized hy-*

percomplex Joukowski transformation of order k. Some geometric mapping properties of this transformation for the special 3D case are accompanied by several images.

Parts of Chapter 2 were already published in:

C. Cruz, M.I. Falcão, and H.R. Malonek. 3D mappings by generalized Joukowski transformations. In B. Murgante et al., editors, *Computational Science and its Applications-ICCSA 2011. Part III, volume 6784 of Lecture Notes in Comput. Sci.*, 358-373. Springer, Heidelberg, 2011.

C. Cruz, M.I. Falcão, and H.R. Malonek. On a quasi-conformal Joukowski type transformation of second order in $\mathbb{R}^{n+1}$. In K. Gũrlebeck, editor, *9th International Conference on Clifford Algebras and their Applications in Mathematical Physics*, Weimar, Germany, 2011.

Chapter 3 relies on the idea that generalized Appell sequences can be used to derive new combinatorial relations. The explicit expression of the Taylor coefficients of the SAP, for an arbitrary dimension n, is obtained. This leads to the consideration of the *Pascal* $n-$*simplex* with hypercomplex entries. The study of various patterns in such structure and the discussion of properties and combinatorial identities involving the aforementioned coefficients is carried out. Finally, the last section of this chapter contains illustrations of the special cases $n = 2$ and $n = 3$.

The particular case of the Pascal tetrahedron was discussed in:

C. Cruz, M.I. Falcão, and H.R. Malonek. About Pascal's tetrahedron with hypercomplex entries. In T. Simos, G. Psihoyios, and Ch. Tsitouras, editors, *AIP Conference Proceedings*, volume 1558, 509-512, Rhodes, Greece, 2013.

In **Chapter 4** we study the concept of *totally regular variables* (TRV), originally introduced by Delanghe in [47] and later on considered by Gürlebeck [75] in quaternionic context. A TRV is a linear hypercomplex holomorphic function whose integer powers are also holomorphic. Under some natural normalization condition, the set of all TRV with values in $\mathbb{R}^3$ is characterized. This leads to the construction of TRV in arbitrary dimensions that are isomorphic to the complex variable $z = x_0 + ix_1$. Among several methods for generating holomorphic functions defined in $\mathbb{R}^{n+1}$ there are some that consist of an appropriate substitution of variables in complex valued holomorphic functions. The method we choose, first mentioned in [43], leads to power functions isomorphic to the classic complex powers z^k which are obtained as powers of TRV. Naturally, they satisfy the Appell property. For all these reasons we call those polynomials *pseudo-complex powers* (PCP). In the end we extend a result of [58] by considering the case of four variables. Besides the Appell sequences of PCP and SAP, a new Appell sequence was detected.

These topics have been discussed in:

C. Cruz, M.I. Falcão, and H.R. Malonek. On the structure of generalized Appell sequences of paravector valued homogeneous monogenic polynomials. In T. Simos, G. Psihoyios, and Ch. Tsitouras, editors, *AIP Conference Proceedings*, volume 1479, 283-287, Kos, Greece, 2012.

C. Cruz, M.I. Falcão, and H.R. Malonek. A note on totally regular variables and Appell sequences in hypercomplex function theory. In B. Murgante et al., editors, *Computational Science and its Applications—ICCSA 2013. Part I*, volume 7971 of *Lecture Notes in Comput. Sci.*, 293-303. Springer, 2013.

C. Cruz, M.I. Falcão, and H.R. Malonek. Monogenic polynomials of four variables with binomial expansion. In B. Murgante et al., editors, *Computational Science and its Applications—ICCSA 2014. Part I*, volume 8579 of *Lecture Notes in Comput. Sci.*, 204-220. Springer, 2014.

Chapter 5 is organized as follows. We start by deducing the necessary and sufficient conditions for characterizing a set of PCP as a basis for the $\mathbb{H}-$linear space $\mathcal{M}_k(\mathbb{H},\mathbb{H})$ of homogeneous holomorphic $\mathbb{H}-$valued polynomials of degree k. For the $3D-$case this requires the choice of a *parameter set* of vectors in S^2, which is naturally related to the primitive roots of unity in the complex plane. The freedom in choosing a certain parameter set allows a variety of approaches depending on the considered application. A particular problem that we are discussing is the expression of all GP in $\mathcal{M}_k(\mathbb{H},\mathbb{H})$ by PCP. This leads to the consideration of Vandermonde matrices and their inversion. It allows later to obtain the Taylor series expansion of a holomorphic function in terms of PCP.

The last part of the chapter is devoted to a detailed discussion of numerical aspects of the PCP. We propose an efficient algorithm for constructing an arbitrary basis of PCP. Some numerical comparisons show that, from the computational point of view, PCP can be a good alternative to the GP, since they can be computed by a less time-consuming algorithm. Finally, we present a recursive algorithm, inspired by the work of Eisinberg-Fedele [52], for inverting Vandermonde matrices for a particular set of nodes.

Some of the topics in this chapter were already published in:

C. Cruz, M.I. Falcão, and H.R. Malonek. On pseudo-complex basis for monogenic polynomials. In S. Sivasundarum, editor, *AIP Conference Proceedings*, volume 1493, 350-356, Vienna, Austria, 2012.

C. Cruz, M.I. Falcão, and H.R. Malonek. On numerical aspects of pseudo-complex powers in $\mathbb{R}^3$. In B. Murgante et al., editors, *Computational Science and its Applications—ICCSA 2014. Part I*, volume 8579 of *Lecture Notes in Comput. Sci.*, 1-16. Springer, 2014.

Finally, **Chapter 6** is concerned with new combinatorial aspects of generalized Appell polynomials that arise when we express, in the case of three real variables, the SAP as a linear combination of PCP. The obtained relationship between these two types of holomorphic Appell polynomials has naturally led to identities of two types: sums of even powers of *cosine* and a Riordan-Sofo type binomial identity. In most of them, an interesting sequence of numbers which involves the central binomial coefficients plays an important role.

The main results of this chapter, proved by different methods, are contained in:

C. Cruz, M.I. Falcão, and H.R. Malonek. Monogenic pseudo-complex power functions and their applications. *Math. Methods Appl. Sci.*, 37:1723-1735, 2014.

Chapter 1

Clifford Analysis toolbox

1.1 The Clifford Algebra $\mathcal{C}\ell_{0,n}$

There should be a branch of mathematics which in a purely abstract way produces similar laws as they appear linked to space in geometry.
H. G. Grassman.

Clifford algebras as associative noncommutative real algebras can be defined in several ways. Here we introduce the basis of the special finite-dimensional Clifford Algebra $\mathcal{C}\ell_{0,n}$ by means of an orthonormal basis of the real Euclidean space $\mathbb{R}^n$. Thus, let $\{e_1, e_2, \ldots, e_n\}$ be an orthonormal basis of $\mathbb{R}^n$ subject to the multiplication rules

$$e_i e_j = -e_j e_i, \quad i \neq j, \quad i, j = 1, \ldots, n, \tag{1.1}$$

$$e_i^2 = -1. \tag{1.2}$$

The associative universal Clifford Algebra $\mathcal{C}\ell_{0,n}$ over $\mathbb{R}$ is the set of numbers $z \in \mathcal{C}\ell_{0,n}$ written in the form

$$z = \sum_A z_A e_A, \tag{1.3}$$

where z_A are real numbers and the basis $\{e_A : A \subseteq \{1, \ldots, n\}\}$ is formed by

$$e_A = e_{h_1} \ldots e_{h_r}, \qquad 1 \leq h_1 < \cdots < h_r \leq n, \qquad e_\emptyset = e_0 = 1. \tag{1.4}$$

Each element of the form (1.3) is usually called ***Clifford number*** or ***hypercomplex number*** (see [76, 105, 139, 140]).

Observe that any product $e_{t_1}\dots e_{t_s} \in \mathcal{C}\ell_{0,n}$ can be rearranged into the standard form of a basis element (1.4). By using (1.1) one can easily obtain $e_{t_1}\dots e_{t_s} = \pm e_{h_1}\dots e_{h_r}$ with $h_1 < \cdots < h_r$ and $r \leq s$.

For each $k = 0, \dots, n$, we use the real vector subspace of $\mathcal{C}\ell_{0,n}$

$$\mathcal{C}\ell_{0,n}^k = \mathrm{span}_{\mathbb{R}}\{e_A : \ |A| = k\} \tag{1.5}$$

which has dimension $\binom{n}{k}$ and whose elements are designated by *k-vectors*. In addition, any linear combination of $k-$vectors is also called a $k-$vector. From (1.5) we get

$$\mathcal{C}\ell_{0,n} = \bigoplus_{k\leq n} \mathcal{C}\ell_{0,n}^k \tag{1.6}$$

and the projection of $z \in \mathcal{C}\ell_{0,n}$ into $\mathcal{C}\ell_{0,n}^k$ is given by

$$[z]_k = \sum_{|A|=k} z_A e_A.$$

Based on this any Clifford number can be written in the form

$$z = [z]_0 + [z]_1 + \cdots + [z]_n$$

and the dimension of $\mathcal{C}\ell_{0,n}$ is calculated by

$$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n. \tag{1.7}$$

Remark 1.1.1 If $n = 0$ then $\mathcal{C}\ell_{0,0}$ is the algebra $\mathbb{R}$ of real numbers. Identifying e_1 with the imaginary unit i, the Clifford Algebra $\mathcal{C}\ell_{0,1}$ is isomorphic to the algebra $\mathbb{C}$ of complex numbers. Finally, $\mathcal{C}\ell_{0,2}$ is isomorphic to the algebra $\mathbb{H}$ of real quaternions (Hamilton's quaternions) by identifying e_1, e_2 and e_1e_2 with the basis quaternions i, j and k, respectively. Often Hamilton's quaternions are also constructed as the even subalgebra of the Clifford Algebra $\mathcal{C}\ell_{0,3}$ identifying, for example, e_2e_3, e_3e_1 and e_1e_2 with the basis quaternions i, j and k, respectively (see [76]). ♦

Because our interest lies on functions defined in $\mathbb{R}^{n+1}$ with values in $\mathbb{R}^{n+1}$ by using the Clifford Algebra $\mathcal{C}\ell_{0,n}$, we will consider two different types of hypercomplex structures of $\mathbb{R}^{n+1}$, both presented in detail in [101]. Each of them may be interpreted as higher dimensional *complexification* of the real Euclidean space $\mathbb{R}^{n+1}$ similar to the well known association of $\mathbb{R}^2$ with $\mathbb{C}$.

The first approach is based on the identification of a real $(n+1)-$tuple with *one* element of the algebra.

Definition 1.1.2 *A* ***paravector*** x *is an element of* $\mathcal{A}_n = \mathrm{span}_{\mathbb{R}}\{e_1, \ldots, e_n\} = \mathcal{C}\ell_{0,n}^0 \bigoplus \mathcal{C}\ell_{0,n}^1$ *of the form*

$$x = x_0 + \underline{x} = x_0 + x_1 e_1 + \cdots + x_n e_n.$$

In this case, $\mathbb{R}^{n+1}$ is embedded in $\mathcal{C}\ell_{0,n}$ by the identification of the vector in $\mathbb{R}^{n+1}$

$$(x_0, \vec{x}) = (x_0, x_1, \ldots, x_n)$$

with the paravector $x \in \mathcal{A}_n \subset \mathcal{C}\ell_{0,n}$.

The ***conjugate*** of a paravector $x \in \mathcal{A}_n$ is defined by

$$\bar{x} = x_0 - \underline{x}.$$

Instead of the real and the imaginary parts we will distinguish between the ***scalar part*** of x

$$\mathrm{Sc}(x) = x_0 = \frac{1}{2}(x + \bar{x})$$

and the ***vector part*** of x

$$\mathrm{Vec}(x) = \underline{x} = e_1 x_1 + \cdots + e_n x_n = \frac{1}{2}(x - \bar{x}).$$

The ***norm*** of x is, like in the complex case, the square root of the product of x and $\bar{x}$, i.e.

$$|x| = (x\bar{x})^{\frac{1}{2}} = (x_0^2 + x_1^2 + \cdots + x_n^2)^{\frac{1}{2}}$$

and it immediately follows that each non-zero paravector x has an inverse given by

$$x^{-1} = \frac{\bar{x}}{|x|^2}.$$

The usual approach to hypercomplex function theory considers $\mathcal{C}\ell_{0,n}$-valued functions of the form $f(x) = \sum_A f_A(x) e_A, \;\; f_A(x) \in \mathbb{R}$, as mappings

$$f : \Omega \subset \mathbb{R}^{n+1} \cong \mathcal{A}_n \longmapsto \mathcal{C}\ell_{0,n}.$$

The big advantage of this approach is to deal with only one hypercomplex variable x. Compared with the ordinary case of two real and one complex variables ($\mathbb{R}^2 \cong \mathbb{C}$), it reflects the reduction of the real dimension $n+1$ to dimension 1.

A second approach concerns the use of the so-called hypercomplex ***Fueter variables***, which are defined as

$$z_k = -\frac{1}{2}(x e_k + e_k x) = x_k - x_0 e_k, \quad k = 1, \ldots n. \tag{1.8}$$

In accordance to [101], a second hypercomplex structure of $\mathbb{R}^{n+1}$ different from that given by $\mathcal{A}_n$ consists in the following isomorphism:

$$\mathbb{R}^{n+1} \cong \mathcal{H}^n = \{\vec{z} : \ \vec{z} = (z_1, \ldots, z_n)\}.$$

This means to take n copies $\mathbb{C}_k$ of $\mathbb{C}$ identifying $i \cong e_k$, $(k = 1, \ldots, n)$, $x_0 \cong \Re z$, $x_k \cong \Im z$, where $z \in \mathbb{C}$ and $\mathbb{C}_k = -e_k\mathbb{C}$. Then $\mathcal{H}^n$ is the cartesian product $\mathcal{H}^n = \mathbb{C}_1 \times \cdots \times \mathbb{C}_n$ and $\mathcal{C}\ell_{0,n}$-valued functions $f(z) = \sum_A f_A(z)e_A$ are considered as mappings

$$f : \Omega \subset \mathbb{R}^{n+1} \cong \mathcal{H}^n \longmapsto \mathcal{C}\ell_{0,n}.$$

We note that for any $\vec{z} = (z_1, \ldots, z_n)$, $\vec{w} = (w_1, \ldots, w_n) \in \mathcal{H}^n$, with $z_k = x_k - x_0 e_k$ and $w_k = y_k - y_0 e_k$, and $\lambda \in \mathbb{R}$ we have that

$$\vec{z} + \vec{w} = (z_1 + w_1, \ldots, z_n + w_n) \in \mathcal{H}^n$$

and

$$\lambda\vec{z} = (\lambda z_1, \ldots, \lambda z_n) \in \mathcal{H}^n,$$

i.e. $\mathcal{H}^n$ is a (right and left) submodule of $\left(\mathcal{C}\ell_{0,n}\right)^n$ over $\mathbb{R}$. Moreover, it is possible to define in $\mathcal{H}^n$ the following hermitian inner product

$$< \vec{z}, \vec{w} >= \sum_{k=1}^{n} z_k \overline{w_k} = \Big(n x_0 y_0 + \sum_{k=1}^{n} x_k y_k\Big) + \sum_{k=1}^{n} (x_k y_0 - x_0 y_k) e_k \ \in \mathcal{A}_n, \tag{1.9}$$

which induces the norm

$$||\vec{z}|| = \sqrt{< \vec{z}, \vec{z} >} = \Big(n x_0^2 + \sum_{k=1}^{n} x_k^2\Big)^{\frac{1}{2}}.$$

Thus, although we do not obtain an isometric relation between $\mathbb{R}^{n+1}$ and $\mathcal{H}^n$, the inequality

$$\frac{1}{\sqrt{n}} ||\vec{z}|| \leq |x| \leq ||\vec{z}||$$

allows to establish the equivalence between the norms $|\cdot|$ and $||\cdot||$ (cf. [100]).

1.2 Clifford holomorphic functions

We focus now on functions with values in the Clifford Algebra $\mathcal{C}\ell_{0,n}$ which have properties similar to those of complex valued holomorphic functions. In this sense, we recall briefly some concepts associated with holomorphic functions in $\mathbb{C}$, following the explanations in [76].

Let $x = x_0 + i x_1 \in \mathbb{C}$ and $f = f(x) = u(x) + i v(x)$ a complex-valued function defined in a domain $\Omega \subset \mathbb{C}$ such that u and v are continuously real differentiable functions. Then

$f \in C^1(\Omega)$ and the differential of f can be written as

$$df = (u_{x_0} + iv_{x_0})dx_0 + (u_{x_1} + iv_{x_1})dx_1, \tag{1.10}$$

where u_{x_k}, v_{x_k} are the partial derivatives of u and v with respect to $x_k,\ k = 0, 1$. We also consider the linear partial differential operators of first order which are generally known by Wirtinger's derivatives

$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x_0} + i\frac{\partial}{\partial x_1}\right) \quad \text{and} \quad \frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x_0} - i\frac{\partial}{\partial x_1}\right), \tag{1.11}$$

together with

$$dx_0 = \frac{1}{2}(dz + d\bar{z}), \quad dx_1 = \frac{1}{2i}(dz - d\bar{z}).$$

Therefore, it follows that (1.10) can be written as

$$df = \big(u_z + iv_z\big)dz + \big(u_{\bar{z}} + iv_{\bar{z}}\big)d\bar{z} = f_z dz + f_{\bar{z}} d\bar{z}. \tag{1.12}$$

The first approach to the holomorphy concept has to due with the approximation of f by complex valued linear functions. In concrete, a function $f \in C^1(\Omega)$ defined in a domain $\Omega \subset \mathbb{C}$ is called holomorphic, if for each $z_0 \in \Omega$ a complex number $f'(z_0)$ exists, such that

$$f(z_0 + h) = f(z_0) + f'(z_0)h + o(h), \tag{1.13}$$

with $o(h) \to 0$ when $h \to 0$. Of course, this is equivalent to say that $f'(z_0)$ can be determined by

$$f'(z_0) = \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}, \tag{1.14}$$

whenever the limit exists. In both cases, the number $f'(z_0)$ is called the complex derivative of f at z_0.

Additionally, holomorphy can also be defined in terms of the well known differential Cauchy-Riemann (CR) equations expressed in the following

Theorem 1.2.1 *A function $f \in C^1(\Omega)$ is holomorphic in Ω if and only if*

$$u_{x_0} - v_{x_1} = 0, \quad u_{x_1} + v_{x_0} = 0, \tag{1.15}$$

or equivalently,

$$\frac{\partial f}{\partial \bar{z}} = f_{\bar{z}} = 0. \tag{1.16}$$

As an immediate consequence of (1.12), it follows that any holomorphic function f does not depend on $\bar{z}$ and that

$$\frac{df}{dz} = \frac{\partial f}{\partial z} = \frac{\partial f}{\partial x_0} = -i\frac{\partial f}{\partial x_1}. \tag{1.17}$$

We turn back our attention to $\mathcal{C}\ell_{0,n}$-valued functions and adopt the following definition which can be seen as the hypercomplex version of (1.13) (cf. [105]).

Definition 1.2.2 *Let $f \in C^1(\Omega)$ be a function defined in a domain $\Omega \subset \mathcal{H}^n \cong \mathbb{R}^{n+1}$, having values in $\mathcal{C}\ell_{0,n}$. The function f is called* ***left-Clifford holomorphic*** *at $\vec{z} \in \Omega$ (resp.* ***right-Clifford holomorphic****) if there exist n Clifford numbers A_k depending on $\vec{z}$, such that*

$$f(\vec{z} + \Delta\vec{z}) = f(\vec{z}) + \Delta z_1 A_1 + \cdots + \Delta z_n A_n + o(||\Delta\vec{z}||),$$

$\Delta\vec{z} = (\Delta z_1, \ldots, \Delta z_n)$ such that $\Delta z_k = h_k - h_0 e_k$ and

$$\lim_{\Delta\vec{z}\to 0} \frac{o(||\Delta\vec{z}||)}{||\Delta\vec{z}||} = 0.$$

In [77] the authors proved that a $\mathcal{C}\ell_{0,n}-$valued function is holomorphic if and only if it belongs to the kernel of a linear partial differential operator which generalizes the Wirtinger's derivative with respect to $\bar{z}$ (recall (1.11)). Indeed, if we set

$$\partial_0 = \frac{\partial}{\partial x_0}, \qquad \partial_{\underline{x}} = e_1\frac{\partial}{\partial x_1} + \cdots + e_n\frac{\partial}{\partial x_n},$$

then the generalization of the Wirtinger's derivatives is given by the following

Definition 1.2.3 *The operator*

$$\overline{\partial} = \frac{1}{2}(\partial_0 + \partial_{\underline{x}}) \tag{1.18}$$

is called ***generalized Cauchy-Riemann operator*** *and*

$$\partial = \frac{1}{2}(\partial_0 - \partial_{\underline{x}}) \tag{1.19}$$

is called the ***conjugate generalized Cauchy-Riemann operator****.*

The aforementioned result reads as follows.

Theorem 1.2.4 *Let $f = f(\vec{z})$ be continuously real differentiable in an open set $\Omega \subset \mathcal{H}^n$. Then f is left-Clifford holomorphic in Ω (resp. right-Clifford holomorphic), if and only if*

$$\overline{\partial} f = 0 \qquad \textit{in } \Omega \qquad \textit{(resp. } f\overline{\partial} = 0\textit{)},$$

where $\overline{\partial}$ is the generalized CR operator which acts on the function f from the left (resp. right).

We note that, in general, left (resp. right) holomorphic functions are not right (resp. left) holomorphic. Nevertheless, all the results achieved to left holomorphic functions can be adapted to right holomorphic functions.

Throughout this work we will be particularly interested in $\mathcal{A}_n-$valued functions of the paravector variable $x \in \mathcal{A}_n$, i.e. we are going to study functions of the form

$$f(x_0, \underline{x}) = \sum_{j=0}^{n} f_j(x_0, \underline{x}) e_j, \tag{1.20}$$

for some real functions $f_0, \ldots, f_n$. In such cases, the action of the operator $\overline{\partial}$ on f from the left is given by

$$\overline{\partial} f = \frac{1}{2} \sum_{i=0}^{n} \sum_{j=0}^{n} e_i e_j \frac{\partial f_j}{\partial x_i},$$

while its action from the right is given by

$$f \overline{\partial} = \frac{1}{2} \sum_{i=0}^{n} \sum_{j=0}^{n} e_j e_i \frac{\partial f_j}{\partial x_i}.$$

It is worth noting that in this case left holomorphic functions are also right holomorphic. This fact follows easily by direct inspection of the *generalized Riesz systems* (see [134]), i.e. the two real systems of first order partial differential equations corresponding to $\overline{\partial} f = 0$ and $f \overline{\partial} = 0$.

The use of Stokes' theorem allows to write the concept of hypercomplex derivability by a limit of a quotient of two integrals in a way similar to what was considered by Pompeiu in [120]. More precisely, in [77] and [101] it was proved that a Clifford holomorphic function is hypercomplex differentiable in Ω if and only if it has a uniquely defined areolar derivative f' in each point of Ω, in the sense of Pompeiu. Then f is real differentiable and f' is given by

$$f' = \partial f = \frac{1}{2}(\partial_0 - \partial_{\underline{x}}) f. \tag{1.21}$$

Since a holomorphic function f belongs to the kernel of $\overline{\partial}$, it follows immediately that, in fact,

$$f' = \partial_0 f = -\partial_{\underline{x}} f,$$

which shows the analogy with the complex derivative (see (1.17)).

Remark 1.2.5 Historically, $\mathcal{C}\ell_{0,n}$-valued functions in the kernel of $\overline{\partial}$ have been referred to by different names. In [64], Fueter named them *regular* (for the special case of $\mathbb{H}$-valued functions), while Delanghe in [47] has chosen the expression *regular analytic*. One of the most used designations which appears in the famous book [23] is the term *monogenic*. Finally, in the recent book [76] the expression *Clifford holomorphic functions* is applied in order to emphasize that hypercomplex function theory is a natural generalization of one complex variable function theory. Throughout this work we adopt this nomenclature and use, whenever it is clear, the term *holomorphic function*. The complex case will be distinguished by writing $\mathbb{C}-$*holomorphic function*. ♦

Remark 1.2.6 The operator $\partial_{\underline{x}}$ corresponds to the well-known Dirac operator, which allows to factorize the Laplacian in $\mathbb{R}^n$ like in $\mathbb{C}$ (cf. [6]) by

$$-\partial_{\underline{x}}^2 = \Delta_n = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}.$$

Furthermore, it is also possible to factorize the Laplacian in $\mathbb{R}^{n+1}$ as

$$4\overline{\partial}\partial = \Delta_{n+1} = \frac{\partial^2}{\partial x_0^2} + \cdots + \frac{\partial^2}{\partial x_n^2},$$

which allows to conclude that Clifford holomorphic functions and all their real components are also harmonic functions. Often this property, is considered as the essential reason why hypercomplex function theory can be considered as a refinement of Harmonic Analysis. ♦

As first examples of $\mathcal{A}_n-$valued holomorphic functions we indicate the Fueter variables $z_k = x_k - x_0 e_k$ (see (1.8)) and their products $e_k z_k = x_0 + x_k e_k$. Indeed, any function which is obtained by substituting in a $\mathbb{C}$-holomorphic function $f(x_0, x_1) = u(x_0, x_1) + iv(x_0, x_1)$ the variables x_1 and i by x_k and e_k, respectively, is an $\mathcal{A}_n$-valued holomorphic function.

One of the drawbacks of working in a noncommutative algebra is the fact that, in general, the product and the composition of Clifford algebra-valued holomorphic functions are not holomorphic. For example, any product of two distinct Fueter variables, say $z_i z_j$, $(i \neq j)$, is not holomorphic. Moreover, positive integer powers of $x \in \mathcal{A}_n$ are holomorphic only for the case $n = 1$. The next results, which will be needed later on, confirm this assertion.

Lemma 1.2.7 *If* $\underline{x} = x_1 e_1 + \cdots + x_n e_n$ *then*

$$\partial_{\underline{x}}\, \underline{x}^k = \begin{cases} -2s\underline{x}^{2s-1}, & \text{if} \quad k = 2s \\ -(n+2s)\underline{x}^{2s}, & \text{if} \quad k = 2s+1 \end{cases}. \tag{1.22}$$

Proof. Based on $\underline{x}^{2s} = \big(-|\underline{x}|^2\big)^s = (-x_1^2 - \cdots - x_n^2)^s$ we conclude that

$$\partial_{\underline{x}}\, \underline{x}^{2s} = \sum_{i=1}^{n} e_i \partial_i \big(-|\underline{x}|^2\big)^s = -2s\underline{x}^{2s-2} \sum_{i=1}^{n} x_i e_i = -2s\underline{x}^{2s-1}.$$

In turn, since $\underline{x}^{2s+1} = \underline{x}^{2s}\underline{x}$ it follows that

$$\partial_{\underline{x}}\, \underline{x}^{2s+1} = \big(\partial_{\underline{x}}\, \underline{x}^{2s}\big)\underline{x} + \underline{x}^{2s}\Big(\partial_{\underline{x}}\, \underline{x}\Big) = -2s\underline{x}^{2s-1}\underline{x} + \underline{x}^{2s}\sum_{i=1}^{n} e_i^2 = -(n+2s)\underline{x}^{2s}.$$

■

Proposition 1.2.8 *If k is a non-negative integer, then*

$$\overline{\partial}(x_0+\underline{x})^k = \frac{1-n}{2}\sum_{s=0}^{\lfloor\frac{k-1}{2}\rfloor}\binom{k}{2s+1}x_0^{k-(2s+1)}\underline{x}^{2s}. \tag{1.23}$$

Proof. The development of x^k in terms of its scalar and vector part can be written as

$$(x_0+\underline{x})^k = \sum_{s=0}^{\lfloor\frac{k}{2}\rfloor}\binom{k}{2s}x_0^{k-2s}\underline{x}^{2s} + \sum_{s=0}^{\lfloor\frac{k-1}{2}\rfloor}\binom{k}{2s+1}x_0^{k-(2s+1)}\underline{x}^{2s+1}. \tag{1.24}$$

The application of the generalized CR operator to the powers of $x = x_0 + \underline{x}$ gives

$$\overline{\partial}x^k = \overline{\partial}(\mathrm{Sc}(x^k)+\mathrm{Vec}(x^k)) = \frac{1}{2}\big(\partial_0\,\mathrm{Sc}(x^k) + \partial_0\,\mathrm{Vec}(x^k) + \partial_{\underline{x}}\,\mathrm{Sc}(x^k) + \partial_{\underline{x}}\,\mathrm{Vec}(x^k)\big).$$

We now group the terms in this last expression based on the parity of the powers of $\underline{x}$. The odd powers of $\underline{x}$ are all contained in $A = \partial_0\,\mathrm{Vec}(x^k) + \partial_{\underline{x}}\,\mathrm{Sc}(x^k)$ which, by the use of Lemma 1.2.7, is given by

$$\begin{aligned}
A &= \sum_{s=0}^{\lfloor\frac{k-1}{2}\rfloor}\binom{k}{2s+1}(k-2s-1)x_0^{k-2s-2}\underline{x}^{2s+1} + \sum_{s=1}^{\lfloor\frac{k}{2}\rfloor}\binom{k}{2s}(-2s)x_0^{k-2s}\underline{x}^{2s-1}\\
&= \sum_{s=1}^{\lfloor\frac{k-1}{2}\rfloor+1}\binom{k}{2s-1}(k-2s+1)x_0^{k-2s}\underline{x}^{2s-1} - \sum_{s=1}^{\lfloor\frac{k}{2}\rfloor}\binom{k}{2s}(2s)x_0^{k-2s}\underline{x}^{2s-1}.
\end{aligned}$$

If k even it follows at once that $\lfloor\frac{k-1}{2}\rfloor + 1 = \lfloor\frac{k}{2}\rfloor$. In turn, when k is odd the term of the first sum corresponding to $s = \lfloor\frac{k-1}{2}\rfloor + 1$ vanishes and the term corresponding to $s = \lfloor\frac{k-1}{2}\rfloor$ may also be written as $\lfloor\frac{k}{2}\rfloor$. This observation allows to rewrite

$$A = \sum_{s=1}^{\lfloor\frac{k}{2}\rfloor}\left[(k-2s+1)\binom{k}{2s-1} - 2s\binom{k}{2s}\right]x_0^{k-2s}\underline{x}^{2s-1}$$

and consequently, since $(k-2s+1)\binom{k}{2s-1} = 2s\binom{k}{2s}$, we conclude that $A = 0$.

The even powers of $\underline{x}$ are grouped in $B = \partial_0\,\mathrm{Sc}(x^k) + \partial_{\underline{x}}\,\mathrm{Vec}(x^k)$. By use of Lemma 1.2.7 we may write

$$B = \sum_{s=0}^{\lfloor\frac{k}{2}\rfloor}\binom{k}{2s}(k-2s)x_0^{k-2s-1}\underline{x}^{2s} + \sum_{s=0}^{\lfloor\frac{k-1}{2}\rfloor}\binom{k}{2s+1}(-n-2s)x_0^{k-2s-1}\underline{x}^{2s}.$$

An analogous reasoning allows to conclude that

$$\begin{aligned} B &= \sum_{s=0}^{\lfloor \frac{k-1}{2} \rfloor} \left[(k-2s)\binom{k}{2s} - (n+2s)\binom{k}{2s+1} \right] x_0^{k-2s-1} \underline{x}^{2s} \\ &= (1-n) \sum_{s=0}^{\lfloor \frac{k-1}{2} \rfloor} \binom{k}{2s+1} x_0^{k-(2s+1)} \underline{x}^{2s}, \end{aligned}$$

and the result is proved. ■

Remark 1.2.9 A function f in $\mathbb{R}^{n+1}$ satisfying $\overline{\partial} f = \partial f = 0$ plays the role of a *constant* with respect to hypercomplex derivative. For this reason, these type of functions are known in the literature as *generalized constants* or *holomorphic constants.* For instance, any function of the form

$$f(x_0, \underline{x}) = \lambda_1(x_1e_2+x_2e_1)+\lambda_2(x_2e_3+x_3e_2)+\cdots+\lambda_{n-1}(x_{n-1}e_n+x_ne_{n-1})+\lambda_n(x_ne_1+x_1e_n),$$

where $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$, is a generalized constant. Observe that the property $\partial_{\bar{z}} f = \partial_z f = 0$, for complex valued functions f, leads to the constant function $f(z) = f(x+iy) = a, a \in \mathbb{C}$ which obviously does not depend on y. ♦

The problem of obtaining $\mathcal{A}_n$-valued holomorphic functions from $\mathbb{C}-$valued holomorphic functions was first discussed by Fueter in the paper [62] "Analytische Funktionen einer Quaternionenvariablen". In this work Fueter aimed to construct A_2-valued functions by means of a suitable substitution of variables in a $\mathbb{C}-$holomorphic function. In detail:

Theorem 1.2.10 (Fueter's theorem) *Let $f(z) = f(x+iy) = u_0(x,y) + iu_1(x,y)$ be a complex function holomorphic in a domain Ω in the upper complex half-plane $\mathbb{C}^+$. If F denotes the paravector valued function obtained from f by substituting:*

$$x \to x_0, \quad y \to |x_1e_1 + x_2e_2| \quad \textit{and} \quad i \to \frac{x_1e_1 + x_2e_2}{|x_1e_1 + x_2e_2|},$$

then $\Delta F(x)$ is a left and right holomorphic function.

Since $\overline{\partial}(\Delta F) = 0$, it follows immediately that $4\partial\overline{\partial}(\Delta F) = \Delta^2 F = 0$. This means that F is in fact a bi-harmonic function. Fueter had also proposed in the aforementioned paper a version of Theorem 1.2.10 for quaternionic functions. Ever since then, many generalizations of Fueter's theorem have been proposed which are based on the consideration of powers of the Laplacian operator of higher order. In 1957, Sce extended Fueter's result to $\mathcal{A}_n$ in the following way (cf. [128]):

Theorem 1.2.11 (Fueter-Sce theorem) *Let $f(z) = u_0(x,y) + iu_1(x,y)$ be a holomorphic function defined in $\Omega \subset \mathbb{C}^+$ and let F be the function given by*

$$F(x) = u_0(x_0, |\underline{x}|) + \omega(\underline{x})u_1(x_0, |\underline{x}|), \tag{1.25}$$

where $x = x_0 + \underline{x} \in \mathcal{A}_n$ and $\omega(\underline{x}) = \frac{\underline{x}}{|\underline{x}|}$. For n being an odd positive integer, the function $\Delta^{\frac{n-1}{2}} F$ is left and right holomorphic.

More recently, T. Qian went one step further by considering the case when n is an even positive integer (see [122]). Other extensions can be found in [92], [123] and [132].

Remark 1.2.12 The idea behind Fueter's theorem (and its generalizations) is an appropriate substitution of variables in a $\mathbb{C}-$holomorphic function f in order to obtain an $\mathcal{A}_n-$valued function which belongs to the kernel of a certain higher order differential operator. We call attention to the fact that in general the function F itself is not holomorphic. For instance, if we consider $f(z) = (x+iy)^k$, $k \in \mathbb{Z}$, then one obtains $F(x) = (x_0 + \frac{\underline{x}}{|\underline{x}|}|\underline{x}|)^k = (x_0 + \underline{x})^k$ which, according to Proposition 1.2.8 is not holomorphic. Finally, we underline that although F inherits the structure of f, the same is not true (in general) for $\Delta^{\frac{n-1}{2}} F$. In Chapter 4 a different technique for obtaining $\mathcal{A}_n-$valued holomorphic functions from $\mathbb{C}$-holomorphic functions f, preserving in some sense the structure of f, will be proposed and applied. ♦

1.3 Holomorphic homogeneous polynomials and series representation

We adopt here the notations used in [76] to refer to sets of specific polynomials. The space of holomorphic paravector-valued homogeneous polynomials of degree k will be represented by $\mathcal{H}_k^+$, while $\mathcal{H}_k^-$ denotes the space of holomorphic paravector-valued homogeneous functions in $\mathbb{R}^{n+1} \cong \mathcal{H}^n$ of degree $-(k+n)$. The unions of each of these spaces over all k are denoted by $\mathcal{H}^+$ and $\mathcal{H}^-$, respectively. An element P_k of $\mathcal{H}_k^+$ is called an ***inner spherical*** polynomial while $Q_k \in \mathcal{H}_k^-$ is an ***outer spherical*** function of degree k. Finally, the $\mathcal{C}\ell_{0,n}$ space of homogeneous holomorphic $\mathcal{C}\ell_{0,n}$-valued polynomials of degree k will be denoted by $\mathcal{M}_k^n(\mathcal{C}\ell_{0,n}, \mathcal{C}\ell_{0,n})$.

In the previous section it was already visible that noncommutative multiplication in Clifford algebras can cause many difficulties, particularly in the search for $\mathcal{A}_n-$valued holomorphic functions analogous to the complex powers z^k. This difficulty can be overcome by using a symmetric product, as was proposed in [100].

Definition 1.3.1 *Let $V(+,\cdot)$ be a commutative or noncommutative ring and let $v_k \in V$, with $k = 1,\dots,n$. The "$\times$"-product, or equivalently, the* ***symmetric product****, is defined by*

$$v_1 \times v_2 \times \cdots \times v_n = \frac{1}{n!} \sum_{\pi(s_1,\dots,s_n)} v_{s_1} v_{s_2} \cdots v_{s_n} \tag{1.26}$$

where the sum runs over ***all*** *permutations of $(s_1,\dots,s_n)$.*

The symmetric product is distributive with respect to addition and it is permutative, i.e. a permutation of the factors does not change its value, but it is not associative. We point out that for the cases where V is a commutative ring, the symmetric product reduces to the ordinary product.

We adopt here the convention proposed in [102] for the case when a factor v_j occurs μ_j times in (1.26). In such case we write

$$\vec{v}^{\mu} = {v_1}^{\mu_1} \times \cdots \times {v_n}^{\mu_n} = \underbrace{v_1 \times \cdots \times v_1}_{\mu_1} \times \cdots \times \underbrace{v_n \times \cdots \times v_n}_{\mu_n} \tag{1.27}$$

and set parentheses if the powers are understood in the ordinary way. We observe that (1.27) involves the multi-index $\mu = (\mu_1,\dots,\mu_n)$ and recall that $\mu! = \mu_1!\mu_2!\dots\mu_n!$ and $|\mu| = \sum_j \mu_j \geq 0$.

Despite the absence of associativity, it is possible to establish the following recursive formulas (cf. [100]).

Theorem 1.3.2 (General Recursion Formula) *If $\mu = (\mu_1,\mu_2,\dots,\mu_n)$ is a multi-index, then the following formulas hold*

$$\vec{v}^{\mu} = \frac{1}{|\mu|}[\mu_1 v_1({v_1}^{\mu_1-1} \times {v_2}^{\mu_2} \times \cdots \times {v_n}^{\mu_n}) + \cdots + \mu_n v_n {v_1}^{\mu_1} \times {v_2}^{\mu_2} \times \cdots \times {v_n}^{\mu_n-1}], \tag{1.28}$$

$$\vec{v}^{\mu} = \frac{1}{|\mu|}[\mu_1({v_1}^{\mu_1-1} \times {v_2}^{\mu_2} \times \cdots \times {v_n}^{\mu_n})v_1 + \cdots + \mu_n({v_1}^{\mu_1} \times {v_2}^{\mu_2} \times \cdots \times {v_n}^{\mu_n-1})v_n]. \tag{1.29}$$

In addition, the symmetric product allows to derive the following generalized multinomial formula (cf. [100]).

Theorem 1.3.3 (Generalized Multinomial Formula) *If $v_1,\dots,v_n \in V$, then*

$$(v_1 + v_2 + \cdots + v_n)^k = \sum_{|\mu|=k} \binom{k}{\mu} \vec{v}^{\mu} \tag{1.30}$$

where

$$\binom{k}{\mu} = \frac{k!}{\mu!} = \frac{k!}{\mu_1!\dots\mu_n!}, \qquad k \in \mathbb{N}_0.$$

In particular, for $\alpha, \beta \in \mathbb{R}$ and $n = 2$ one has the binomial formula

$$(\alpha v_1 + \beta v_2)^k = \sum_{s=0}^{k} \binom{k}{s} \alpha^{k-s} \beta^s v_1^{k-s} \times v_2^s. \tag{1.31}$$

If, in convention (1.27) we substitute $\vec{v} = (v_1, \ldots, v_n) \in V^n$ by the $n-$tuple of Fueter variables $\vec{z} = (z_1, \ldots, z_n) \in \mathcal{H}^n$ then we obtain the following $\mathcal{C}\ell_{0,n}-$valued functions which will play an important role in this work.

Definition 1.3.4 *Let $\vec{z} \in \mathcal{H}^n$ and $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{N}_0^n$. The functions*

$$\vec{z}^{\mu} = z_1^{\mu_1} \times \cdots \times z_n^{\mu_n} \tag{1.32}$$

*are called **generalized powers** (GP).*

Each generalized power $\vec{z}^{\mu}$ is an homogeneous polynomial of degree $|\mu|$ which can be considered as a generalization of powers of several complex variables and, consequently, a generalization of the ordinary complex powers z^k (n=1).

The use of (1.26), (1.28) or (1.29) allows to write, for example,

$$\begin{aligned}
z_1 \times z_2 &= \frac{1}{2}(z_1 z_2 + z_2 z_1), \\
z_1^2 \times z_2 &= \frac{1}{3}(z_1^2 z_2 + z_1 z_2 z_1 + z_2 z_1^2), \\
z_1 \times z_2^2 \times z_3 &= \frac{1}{12}(z_1 z_2^2 z_3 + z_1 z_2 z_3 z_2 + z_1 z_3 z_2^2 + z_2 z_1 z_2 z_3 + z_2 z_1 z_3 z_2 + z_2^2 z_1 z_3 \\
&\quad + z_2^2 z_3 z_1 + z_2 z_3 z_1 z_2 + z_2 z_3 z_2 z_1 + z_3 z_1 z_2^2 + z_3 z_2 z_1 z_2 + z_3 z_2^2 z_1).
\end{aligned}$$

Following again [100], we list some important properties of GP.

Proposition 1.3.5 *Consider the GP $\vec{z}^{\mu}$ defined in* (1.32)*, with $\vec{z} \in \mathcal{H}^n$, $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{N}_0^n$ and $|\mu| = k$. Then*

1. *$\vec{z}^{\mu} \in \mathcal{A}_n$;*
2. *$\vec{z}^{\mu}$ are left and right holomorphic;*
3. *The number of GP of degree k which are left and right $\mathcal{C}\ell_{0,n}$-linearly independent is exactly*

$$\binom{n+k-1}{k}.$$

Sudbery proved in [135] that the dimension of the $\mathbb{H}$ space of homogeneous holomorphic $\mathbb{H}-$valued polynomials of degree k is $k+1$. Therefore, if we recall the isomorphism $\mathbb{H} \cong \mathcal{C}\ell_{0,2}$ and use the shorter notation $\mathcal{M}_k(\mathbb{H}, \mathbb{H}) = \mathcal{M}_k^2(\mathcal{C}\ell_{0,2}, \mathcal{C}\ell_{0,2})$ then, according to the previous result, the set

$$\{z_1^k, z_1^{k-1} \times z_2, \ldots, z_1 \times z_2^{k-1}, z_2^k\} \tag{1.33}$$

is a basis for $\mathcal{M}_k(\mathbb{H}, \mathbb{H})$.

These results lead to the introduction of the concept of multiple hypercomplex power series in [102]. In particular, the author considered the following right-, resp. left-, multiple powers series ordered by its degree of homogeneity and centered at the origin:

$$P(\vec{z}) = \sum_{k=0}^{\infty} \Big(\sum_{|\mu|=k} c_\mu \vec{z}^{\,\mu} \Big) \tag{1.34}$$

$$P(\vec{z}) = \sum_{k=0}^{\infty} \Big(\sum_{|\mu|=k} \vec{z}^{\,\mu} c_\mu \Big), \tag{1.35}$$

where $c_\mu \in \mathcal{C}\ell_{0,n}$. In addition, he proved that if each multiple series converges in some polycylindrical domain of the form

$$V(\vec{r}) = \{\vec{z} \in \mathcal{H}^n : |z_k| < r_k, \ k = 1, \ldots, n\}, \tag{1.36}$$

where $\vec{r} \in \mathbb{R}^n$ is such that $r_k \geq 0$, then it generates, inside the domain of convergence, a holomorphic function which coincides with its Taylor series.

Theorem 1.3.6 *Every convergent right-, respectively, left-power series generates in the interior of its domain of convergence a holomorphic function $f(\vec{z})$ which coincides there with its Taylor series, i.e. in a neighborhood of $\vec{z} = \vec{0}$ we have*

$$f(\vec{z}) = \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{|\mu|=k} \frac{\partial^{|\mu|} f(\vec{0})}{\partial \vec{x}^{\mu}} \vec{z}^{\mu}. \tag{1.37}$$

On the other hand, if we consider now $\vec{w} = (w_1, \ldots, w_n) \in \mathbb{C}^n$ and the same multi-index $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{N}_0^n$, then $\vec{w}^\mu = w_1^{\mu_1} \ldots w_n^{\mu_n}$ and the Taylor series expansion of an analytic function f of n complex variables $w_1, \ldots, w_n$ can be written as ([93])

$$f(\vec{w}) = \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{|\mu|=k} \frac{\partial^{|\mu|} f(\vec{0})}{\partial \vec{w}^{\mu}} \vec{w}^{\mu}, \tag{1.38}$$

where f is defined in the polycylindrical domain

$$U(\vec{w}) = \{(w_1, \ldots, w_n) \in \mathbb{C}^n : |w_j| < r_j,\, r_j > 0,\, j = 1, \ldots, n\}.$$

Both expansions (1.37) and (1.38) show the deep analogy between functions of n hypercomplex variables and those of n complex variables. The only one formal difference of both Taylor series expansions is the use of the symmetric product in (1.37) instead of the ordinary complex product in (1.38).

It is also possible to calculate the radius of convergence of any multiple hypercomplex series of the form (1.34) and (1.35) by means of the well known Cauchy-Hadamard formula (cf. [100]).

Theorem 1.3.7 *Let $M_k = \max\{|c_\mu| : \ |\mu| = k\}$ for $k = 0, 1, \ldots,$ then the radius of convergence of the series* (1.34) *(or equivalently* (1.35)*) is given by*

$$R = \Big(\overline{\lim_{k \to +\infty}} \sqrt[k]{M_k} \Big)^{-1}. \tag{1.39}$$

The set $\{\vec{z} \in \mathcal{H}^n : |\vec{z}| < R\}$ is called the domain of convergence of the series. In [23] and [76] one can find many results which generalize some well known properties of complex-valued convergent power series.

Cauchy-Kowalewskaya extension

Many efforts have been made in order to develop new techniques for generalizing to higher dimensional spaces results of complex function theory. Fueter's theorem, together with its generalizations, is an example of how to obtain $\mathcal{A}_n$-valued holomorphic functions based on $\mathbb{C}-$valued holomorphic ones by means of an appropriate change of variables (recall Theorem 1.2.10). Another important technique is the so-called Cauchy-Kowalewskaya (CK) extension. The classical idea behind the concept of CK-extension is to characterize solutions of suitable (systems of) PDE's by their restriction to a submanifold of codimension one (cf. [20] and [32]). The CK theorem in the complex plane concerns the relation between the analyticity domain of a real analytic function and the domain of its holomorphic extension in the complex domain ([24]). In [20], [23] and [100] one can find analogous results which relate analytic functions in $\mathbb{R}^n$ with holomorphic functions with values in $\mathcal{A}_n \cong \mathbb{R}^{n+1}$. After considering the domain (1.36) and the parallelepiped

$$R(\vec{r}) = \{\vec{x} \in \mathbb{R}^n : \ |x_k| < r_k,\ k = 1, \ldots, n\},$$

they can be summarized in the following

Theorem 1.3.8 (CK extension in $\mathcal{C}\ell_{0,n}$) *Let $\vec{x} \in \mathbb{R}^n$, $\vec{z} \in \mathcal{H}^n$ and $\vec{f}$ be real-analytic in $R(\vec{r})$, i.e.*

$$\vec{f}(\vec{x}) = \sum_{k=0}^{+\infty} \frac{1}{k!} \sum_{|\mu|=k} \frac{\partial^{|\mu|} \vec{f}(\vec{0})}{\partial \vec{x}^{\mu}} \vec{x}^{\mu}, \tag{1.40}$$

then the CK extension of $\vec{f}$ to a right-, respectively left-, holomorphic function in $V(\vec{r})$ is given in a unique way by the function

$$f_R^*(\vec{z}) = \sum_{k=0}^{+\infty} \frac{1}{k!} \sum_{|\mu|=k} \frac{\partial^{|\mu|} \vec{f}(\vec{0})}{\partial \vec{x}^{\mu}} \vec{z}^{\mu}, \tag{1.41}$$

$$f_L^*(\vec{z}) = \sum_{k=0}^{+\infty} \frac{1}{k!} \sum_{|\mu|=k} \vec{z}^{\mu} \frac{\partial^{|\mu|} \vec{f}(\vec{0})}{\partial \vec{x}^{\mu}}. \tag{1.42}$$

Furthermore, the restriction of f_R^* and f_L^* to the hyperplane $x_0 = 0$ coincides with $\vec{f}(\vec{x})$.

The functions f_R^* and f_L^* are called the right (resp. left) Cauchy-Kowalewskaya of $\vec{f}$. As an immediate consequence of the previous result one has that both the left and right CK extensions of x_k coincide with the Fueter variable $z_k = x_k - x_0 e_k,\ k = 1, \ldots, n$.

It is also possible to define a CK product by means of the CK extension which allows to construct generalizations of elementary functions of a complex variable in $\mathcal{A}_n$(see [23]).

1.4 Clifford Algebra based software

Modern computer hardware and software have greatly enhanced research in several areas of knowledge, including in applied mathematics and its pure and theoretical foundations.

Despite the context of this work, we have to underline the attention of the computer science community in the algebra of the real quaternions due to the well known relation of quaternions with 3D rotations (see e.g. [9]). In addition, the increasing interest in using general Clifford algebras and their applications in almost all applied sciences has motivated the emergence of several software packages to perform computations, not only in the algebra of real quaternions, but also in more general algebras (see [3, 4] and the references therein for details). For these reasons there are nowadays several Clifford Algebra based software packages available, among which we would like to highlight the following:

1. `CLICAL` a pioneer Clifford Algebra calculator developed by P. Lounesto and collaborators [99] more than 25 years ago. `CLICAL` is a calculator type computer program for vectors, complex numbers, quaternions, bivectors, spinors and multivectors in Clifford algebras.

2. `CLIFFORD` a *Maple* package for performing computations in Grassmann and Clifford algebras [3, 4]. `CLIFFORD` was developed as a basic tool for all investigations and applications which can be carried out in finite dimensional vector spaces equipped with a quadratic form.

3. `QuaternionAlgebra` a package for *Maple* that makes it possible to perform symbolic manipulations with quaternions. `QuaternionAlgebra` represents quaternions as matrices of order 4, and the package is compatible with the *Maple* `LinearAlgebra` package so that matrices and vectors of quaternions can be manipulated (see [126]).

4. `Quaternion` a toolbox for *Matlab* that extends this system to allow calculation with matrices of quaternions in almost the same way that one calculates with matrices of complex numbers. This is achieved by defining a private type to represent quaternion matrices and overloading of many standard *Matlab* functions. The toolbox supports real and complex quaternions (see [127]).

5. `Quaternions` a standard *Mathematica* package for implementing the algebra of real quaternions. `Quaternions` adds rules to arithmetic operations and extends the definition of many functions. Some extensions of the package are available which add new functionalities to the standard package, for example, the ability of using quaternions as surrogates for rotations or the capability of performing symbolic operations on quaternion-valued functions.

We have chosen for performing the numerical experiments[1] a *Mathematica* and a *Maple* packages, both containing some of the most important definitions and procedures needed for our purpose here. These packages are:

1. `Quat` - a *Maple* package for calculations with quaternions which contains a large section for dealing with polynomial calculations in $\mathbb{R}^3$ and $\mathbb{R}^4$ and their applications. Moreover, the package also bundles an internal polynomial database of generalized powers (recall (1.32)). It has been published on a CD-ROM accompanying the text book [76].

2. `Quaternions` - an extension of the standard *Mathematica* package [59], which endows it with the ability to perform operations on quaternion-valued functions, in particular, on a basic set of polynomials which will be defined in Chapter 2. It also extends the applicability of the original package to paravector-valued and complex-like functions.

[1] All simulations in this work have been performed on a computer with Intel Xeon E5607 4C 2.26 GHz/1066Mhz/8MB processors and 64GB of RAM.

In this work, the use of the aforementioned software packages takes place in two different moments and with different purposes. In Chapter 2 we make use of the visualization capabilities of *Mathematica* to produce 2D- and 3D-plots of complex and quaternionic mappings. In Chapter 5, by taking advantage of the powerful symbolic technology of both systems, different algorithms are implemented in *Maple* and *Mathematica* relying on `Quat` and `Quaternions`, respectively, to manipulate generalized powers and a new family of polynomials, both with values in $\mathbb{R}^3$.

1.5 Historical remarks

This section outlines some basic historical aspects related to the emergence and development of Clifford algebras as well as hypercomplex function theory. It is not our intension to make an exhaustive survey of the history behind these two topics, but rather to give some insight into some main steps which have contributed to these research fields. In parallel, these *Historical Remarks* are in some sense a complement to the first three sections of this chapter.

Mathematical physicist from an early age, W. R. Hamilton (1805-1865) has been involved in several algebraic questions. One of his biggest achievements was on showing that the complex numbers build an algebra with units 1 and i such that $1^2 = 1$ and $i^2 = -1$ and, simultaneously on proving that rotations in a plane are described by the algebra of such numbers. Regarded as one of the great "abstractors" of algebra, Hamilton tried to develop a three-dimensional analogue of complex numbers based on this idea of a *geometric algebra*. From here until the discovery of quaternions the story is widely known. Hamilton had no problem in adding and subtracting triples (which he called *vectors*), but how to multiply them to allow division continued to escape him for a long time. In fact, it took him ten years to overcome this problem and to introduce a third imaginary unit k besides i and j such that $1^2 = 1$ and $i^2 = j^2 = k^2 = ijk = -1$ and to drop commutativity in order to be able to divide vectors. Hamilton had finally discovered the first noncommutative division algebra - the algebra of quaternions, that is, the four-dimensional normed division algebra over the real numbers (see [76] and [116]).

Later on, in 1844, with the work *Die Ausdehnungslehre* which translates as "theory of extension", H. G. Grassmann (1809-1877) showed that once geometry is put into the algebraic form then the dimensional space can be generally described for any dimension greater or equal than three. Heavily influenced by philosophical considerations, he begun with a collection of "units" or "elementary quantities" $e_1, e_2, e_3, \ldots$ and defined the free linear space which they generate. Besides developing the concept of Linear Algebra, Grassmann introduced also a geometric product defined for spaces of arbitrary dimension - *the exterior product*, also known as combinatorial product that would become the key operation of what is nowadays called *exterior algebra* or *Grassmann's algebra*. According to [84], Grassmann dedicated the last years of his life to fit quaternions into extension theory, which led him to the definition of a *central product* $\boldsymbol{ab}$ of vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ by writing

$$\boldsymbol{ab} = \boldsymbol{\lambda}[\boldsymbol{a}|\boldsymbol{b}] + \boldsymbol{\mu}[\boldsymbol{ab}], \tag{1.43}$$

where $[\boldsymbol{a}|\boldsymbol{b}]$ is the inner product, $[\boldsymbol{ab}]$ is the outer (or exterior) product and λ, μ are arbitrary nonzero constants. If we consider $\lambda = \mu = 1$, relation (1.43) can be written in modern notation as

$$\boldsymbol{ab} = \boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{a} \wedge \boldsymbol{b}. \tag{1.44}$$

Surprisingly, (1.44) is essentially the basic product of the "geometric algebras" which were defined later by W. K. Clifford (1845-1879). As it is explained in [84], the lack of time to investigate the central product led him to an absence of a clear conclusion of the adaptation of quaternions into the extension theory. This was done by Clifford in his successful attempt of combining quaternions with Grassmann's extension theory.

The Grassmann exterior algebra has no inner product and does not require a metric usually induced by that product. Therefore Clifford, motivated by Hamilton's ideas, introduced in 1878 a new *geometric product* where metric relations for products of vectors (realized by areas, volumes etc.) also have place (cf. (1.44)). Thus, any Clifford Algebra is generated from a vector space with inner product by an exterior product, or more specifically, it is a graded structure which is constructed from the bases by creating with the exterior product higher order bases. Among several consequences we highlight that geometric entities, such as lines, planes and volumes become basic elements of the algebra and can be manipulated by a rich set of algebraic operators that have a direct geometric significance (see [50, 51, 86]). R. Lipschitz (1832-1903) also discovered Clifford algebras two years after Clifford and independently of him and applied them to the study of orthonormal transformations[2]. Unfortunately, due to the premature death of Clifford and the birth of the vector calculus developed by J. Gibbs (1839-1903) and O. Heaviside (1850-1925), the development of Clifford algebras potential

[2]Up to 1950 the term "Clifford-Lipschitz numbers" was commonly used to refer to what is nowadays called "Clifford numbers".

remained dormant for decades. In fact vector calculus allowed to describe many physical phenomena primarily in the 3 dimensional Euclidean space $\mathbb{R}^3$ which led to many progresses in differential geometry and in the study of partial differential equations. Nevertheless, in its conventional form the use of a cross product didn't allow generalizations of vector calculus to higher dimensions. In some sense we can say that the "reawakening" of Clifford algebras was brought by the theoretical physicists W. Pauli (1900-1958) and P. Dirac (1902-1984) by showing the usefulness of quaternions in quantum physics and in quantum mechanics and quantum electrodynamics, respectively.

Later on, inspired by the diversity of algebras used to express geometric relations and to describe geometric structures (such as complex numbers, quaternions, matrix algebra, vector, tensor and spinor algebras and the algebra of differential forms) D. Hestenes tried to create a "unified mathematical language to help develop and express" the physical world. Thus, in [85], the authors considered Clifford algebras from a new point of view in order to make it a versatile language and a computational tool both for physics and mathematical purposes. The fact is that over the last two decades Clifford algebras have been widely applied in modern areas such as robotic vision, image and signal processing, virtual reality, computer vision among other disciplines (see [22, 49, 87] and references therein).

Around the same time that Pauli and Dirac were using special Clifford algebras in quantum mechanics and quantum electrodynamics, the worldwide known specialist in Number Theory R. Fueter (1880-1950) started the creation of a new discipline. In his thesis Fueter worked on the connection between the concept of *complex multiplication*[3]. This and further research led him to consider the problem of developing new function theoretical methods which could be adapted to the problem of *hypercomplex multiplication* (cf. [104]). Based on this idea Fueter published several papers where he developed a generalized function theory of quaternions (cf. [62, 63, 64, 65]). Besides defining a general concept of quaternionic multiplication, Fueter proposed a definition of "regular" quaternionic functions by means of an analogue of the Cauchy-Riemann equations which led him to close analogues of Cauchy's theorem, Cauchy's integral formula, and the Laurent expansion. In addition, Fueter studied also the problem of obtaining regular functions with values in the Euclidean 3−dimensional space by means of a suitable substitution of variables in a complex valued holomorphic function.

[3]In Fueter's PhD thesis *Der Klassenkörper der quadratischen Körper und die komplexe Multiplikation* (The class field of quadratic fields and complex multiplication) the term *complex multiplication* is used as a bridge between number theory and function theory via the properties of elliptic functions. More precisely, in complex multiplication the role of the unit circle is taken by a suitable elliptic curve. For more details see [104] and [130].

Based on this, Fueter is considered as one of the forerunners and creators of the fundamentals of a new function theory based on Cliford algebras, i.e. *Clifford Analysis*, which is also called as *Hypercomplex Function Theory* (HFT). Besides being convinced that HFT was a "great theory that needs totally new points of view", Fueter also believed that HFT in higher dimensions would prevail upon several complex variables theory (cf. [104]).

Almost at the same time, but independently from Fueter, the Romanian mathematicians G. C. Moisil (1906-1973) and N. Teodorescu (1908-2000) developed a theory of generalized holomorphic functions with values in the Euclidean 3-dimensional space and applied their results to mechanics.

Of course we shall not forget the previous attempts made in order to develop a function theory in noncommutative algebras in an analogous way to the classical function theory of one complex variable. As an example, at the end of the 19th century, by using a generalization of Cauchy's approach in the complex case, G. Scheffers did not succeeded in finding an appropriate differentiability concept (see [129]). Half a century later, the mathematicians N. Krylov and A. Mejlikhzhon worked on the existence of the limit of a difference quotient of $\mathbb{H}$-valued functions defined in $\mathbb{H}$ analogously to (1.14). They proved in [94] and [112] that the definition of a holomorphic function via this process is restricted to a small class of functions, namely the linear functions $f(x) = a + xb$ and $f(x) = a + bx$, for $a, b \in \mathbb{H}$, depending on if h^{-1} is multiplied by the left or right-hand sides, respectively, in the limit of the quotient difference. Thus, these works clearly indicated that one should consider the definition of $\mathbb{H}$-valued holomorphic functions via a different technique.

The inheritance left by Fueter was retaken in the early 1970s by a research Belgium group headed by R. Delanghe. They rehabilitated the study of a theory of generalized holomorphic functions, which they called *monogenic functions*, with values in a general Clifford Algebra. In cooperation with F. Brackx and F. Sommen many contributions based on this theory which distanced itself from the theory of holomorphic functions of several complex variables started to appear. Besides the references [23] and [81], many authors have contributed to the establishment of HFT in a large variety of modern areas.

Despite several attempts to study Clifford holomorphic functions by a corresponding differentiability concept, the problem of characterizing a *well-defined* hypercomplex derivative remained open for a long time. In the well known paper [135], A. Sudbery succeeded in defining the concept of quaternionic differentiability via the Cauchy approach, that is, by considering *analogues* of the Wirtinger's derivatives (1.11) in $\mathbb{H}$. For the case of a general Clifford Algebra we can mention several different approaches. The first one, which was already presented in Section 1.2 and which will be used throughout this work, corresponds to the characterization of a $\mathcal{C}\ell_{0,n}-$valued holomorphic function via Theorem 1.2.4.

The second approach has its fundamentals in the theory of harmonic functions and was developed by H. Leutwiler and S. -L. Eriksson. Similarly to the complex case, where the components of a complex-valued function are a pair of conjugated harmonic functions, the idea is to consider the decomposition of $f \in \mathcal{C}\ell_{0,n}$ in $f = g + he_n,\ g, h \in \mathcal{C}\ell_{0,n-1}$. In this case each component, g, h, is solution of a differential equation involving the Laplacian. In particular g satisfies the Laplace-Beltrami equation associated with the hyperbolic metric, defined on the upper half space $\mathbb{R}^{n+1}_{+}$. A function f is said to be hypermonogenic if it belongs to the kernel of M, where

$$Mf = Df + \frac{n-1}{x_n}\overline{Qf}.$$

In the complex case, i.e. for $n = 1$, the operator M reduces to the Wirtinger's partial derivative with respect to $\bar{z} \in \mathbb{C}$ (see (1.11)). Particularly, we mention the fact that any integer power of x is hypermonogenic. Details about this approach can be found in [54] and [98].

The third approach is related to the connection between Theorem 1.2.11 (and its generalizations) and the concept of *holomorphic Cliffordian functions* which was introduced by G. Laville in the end of 1990's. The underlying idea is that the generalization of the Wirtinger's partial derivative with respect to $\bar{z}$ to higher dimensional spaces of odd dimension should be a higher order differential operator. More concretely, in accordance with G. Laville and I. Ramadanoff, a $\mathcal{C}\ell_{0,2n+1}$-valued function f is said to be holomorphic Cliffordian if it belongs to the kernel of the differential operator $\overline{\partial}\Delta^n$ (see [97] and [124]). Thus, an immediate consequence of Theorem 1.2.11 is that the function $F(x)$ is holomorphic Cliffordian. In other words, for spaces of odd dimension, Fueter's theorem is a technique for obtaining holomorphic Cliffordian functions.

Last but not least we would like to mention another recent approach for linking directly classical function theoretic problems in one complex variable to those in HFT. This approach is based on so-called *slice monogenic functions* introduced by G. Gentili and D. Struppa almost ten years ago in [69] and [70]. As examples for the treatment of regular functions and power series in the sense of *slice monogenic functions* we refer to the paper [67] and the recent books [31] and [68]. One of the book chapters by A. Perroti ([119]) seems to us of some relevance in the context of our subject, because it deals with *meeting points* of *Fueter regularity and slice regularity.*

Chapter 2

Generalized Appell polynomials

2.1 Introduction

One of the already mentioned drawbacks of Clifford Analysis is the fact that pointwise multiplication of $\mathcal{C}\ell_{0,n}$-valued holomorphic functions, as well as their composition, are not algebraically closed in this class of holomorphic functions. This causes serious problems for the use of corresponding formal power series, for the development of a suitable generating function approach to special holomorphic polynomials, or for establishing relations to corresponding hypergeometric functions, etc. It is also the reason why in polynomial approximation in the context of Clifford Analysis almost every problem needs the development of properly adapted polynomial bases (e.g. [17, 23, 27, 30, 55, 102]).

In this chapter we study a class of power-like holomorphic polynomials that behave like power-law functions under the differentiation operation and discuss some of its properties. The main idea goes back to 1880 with the work of P. Appell [11]. It is based on the construction of sequences of polynomials of one real variable, $\big(P_k(x)\big)_{k\geq 0}$, such that $P_0(x) \neq 0$, with the property that P_k has exactly degree k and

$$P'_k(x) = kP_{k-1}(x), \qquad k = 1, 2, \ldots. \tag{2.1}$$

Any such sequence is now called an Appell sequence or an Appell set. The classical Bernoulli, Hermite and Euler polynomials are well-known examples of such sequences.

In addition, in [11], Appell observed that sequences of polynomials verifying (2.1) also satisfy the following binomial type identity

$$P_k(x) = \sum_{s=0}^{k} \alpha_s \binom{k}{s} x^{k-s}, \qquad k = 0, 1, \ldots,$$

with real coefficients α_s, $s = 0, 1, \ldots, k$.

In hypercomplex analysis context, the hypercomplex derivative of a holomorphic function allows the generalization of the concept of Appell sequences (2.1) to higher dimensions. In [55] one can find the following definition.

Definition 2.1.1 *The sequence* $\left(P_k\right)_{k\geq 0}$, *where* $P_k \in \mathcal{H}_k^+$ *is called* ***generalized Appell sequence*** *with respect to* ∂ *if*

$$P_0(x) \equiv 1, \tag{2.2}$$

$$\partial P_k = k P_{k-1}, k = 1, 2, \ldots. \tag{2.3}$$

Nowadays polynomials satisfying conditions (2.2)-(2.3), are of general interest in Clifford Analysis and consequently they have been studied in detail in several papers by different authors and various applications have been considered. We refer, for example, to the works [17, 26, 27, 33, 43, 45, 46, 55, 56, 57, 58, 106, 110].

In this chapter we survey the main results concerning a special type of Appell polynomials which play an important role in the present work. New contributions related to negative powers constructed by means of these polynomials as well as applications to quasi conformal mappings are also considered.

2.2 Special Appell polynomials and their Kelvin transform

In this section we study a special type of Appell polynomials firstly introduced by Falcão and Malonek in [43]. Our aim is to present the main properties of such polynomials within the context of their historical evolution.

Having in mind the idea of using a special direct power series approach for generating holomorphic functions, the authors introduced in [43], special homogeneous holomorphic polynomials of degree k with respect to the hypercomplex variable $x = x_0 + x_1 e_1 + \cdots + x_n e_n$ and its conjugate $\bar{x} = x_0 - x_1 e_1 - \cdots - x_n e_n$, ($n \geq 2$, arbitrary). In other words, those polynomials have the form

$$\mathcal{P}_k^n(x) = \sum_{s=0}^{k} T_s^k(n)\, x^{k-s}\, \bar{x}^s, \tag{2.4}$$

where $T_s^k(n)$ are suitable defined real numbers. The authors determined the explicit form of the unknowns $T_s^k(2)$ in such a way that the polynomials $\mathcal{P}_k^2$ in (2.4) have the Appell property. The extension to an arbitrary dimension (higher than 2) was discussed and applied later on in [55, 107]. The general result can be stated in the following form.

Proposition 2.2.1 *The sequence of polynomials $\mathcal{P}_k^n$ in* (2.4) *forms a generalized Appell sequence in $\mathcal{A}_n$ if and only if*

$$T_s^k(n) = \frac{k!}{n_{(k)}} \frac{\left(\frac{n+1}{2}\right)_{(k-s)} \left(\frac{n-1}{2}\right)_{(s)}}{(k-s)!s!}, \tag{2.5}$$

where $a_{(r)} = \frac{\Gamma(a+r)}{\Gamma(a)}$, $(r=1,2,\dots)$ denotes the Pochhammer symbol with $a_{(0)} = 1$.

We underline the remarkable role played by the numbers $T_s^k(n)$ in expression (2.4), since they allow to obtain both the holomorphy and the Appell properties, based on the use of a linear combination of powers of two non-holomorphic variables x and $\bar{x}$.

Since the polynomials $\mathcal{P}_k^n$ are holomorphic it seems natural to ask for their canonical expression in terms of the Fueter variables (1.8) $z_l = x_l - x_0 e_l$; $l = 1, 2, \dots, n$. Recalling Theorem 1.3.8, it is clear that the Cauchy-Kowalevskaya extension of the restriction of $\mathcal{P}_k^n$ to the hyperplane $x_0 = 0$ should coincide with $\mathcal{P}_k^n$ itself, due to the unicity theorem for the Taylor series of a holomorphic function (see Theorem 1.3.6). More precisely, we have

Proposition 2.2.2 *The polynomials* (2.4) *can be written in terms of generalized powers as*

$$\mathcal{P}_k^n(x) = \mathbf{P}_k(z_1, \dots, z_n) = c_k(n) \sum_{|\nu|=k} z_1^{\nu_1} \times \cdots \times z_n^{\nu_n} \binom{k}{\nu} e_1^{\nu_1} \times \cdots \times e_n^{\nu_n}, \tag{2.6}$$

where

$$c_k(n) = \sum_{s=0}^{k} (-1)^s T_s^k(n). \tag{2.7}$$

Based on the properties of $\mathcal{P}_k^n$ it is possible to obtain the following identities between the coefficients $T_s^k(n)$ and $c_k(n)$.

Proposition 2.2.3 *The real numbers $T_s^k(n)$ and $c_k(n)$ defined by* (2.5) *and* (2.7), *respectively, have the following properties.*

$$\sum_{s=0}^{k} T_s^k(n) = 1, \tag{2.8}$$

$$\sum_{s=0}^{k} (-1)^s T_s^k(n) = c_k(n) = \Big[\sum_{|\nu|=k} (-1)^k \binom{k}{\nu} \left(e_1^{\nu_1} \times \cdots \times e_n^{\nu_n}\right)^2 \Big]^{-1}. \tag{2.9}$$

Moreover, $c_k(n)$ can be defined recursively as

$$c_k(n) = \begin{cases} \frac{k!!(n-2)!!}{(n+k-1)!!} & \text{if } k \text{ is odd} \\ c_{k-1}(n) & \text{if } k \text{ is even} \end{cases}, \tag{2.10}$$

with $c_0(n) = 1$, for $n \geq 0$. As usual we define $(-1)!! = 0!! = 1$.

The following useful representations of $\mathcal{P}_k^n$ can be obtained by means of straightforward calculations.

Proposition 2.2.4 *The sequence of polynomials $\mathcal{P}_k^n(x)$ admit the following representations*

1. *In terms of x_0 and $\underline{x}$:*

$$\mathcal{P}_k^n(x_0,\underline{x}) = \sum_{s=0}^{k} c_s(n)\binom{k}{s} x_0^{k-s}\underline{x}^s. \tag{2.11}$$

2. *In terms of scalar and vector part:*

$$\mathcal{P}_k^n(x) = u(x_0,|\underline{x}|) + \omega(x)v(x_0,|\underline{x}|),$$

where $\omega(x) = \frac{\underline{x}}{|\underline{x}|}, \underline{x} \neq 0$, and u and v are the real valued functions

$$u(x_0,|\underline{x}|) = \sum_{s=0}^{\lfloor \frac{k}{2} \rfloor} \binom{k}{2s}(-1)^s c_{2s}(n) x_0^{k-2s}|\underline{x}|^{2s} \tag{2.12}$$

$$v(x_0,|\underline{x}|) = \sum_{s=0}^{\lfloor \frac{k-1}{2} \rfloor} \binom{k}{2s+1}(-1)^s c_{2s+1}(n) x_0^{k-2s-1}|\underline{x}|^{2s+1}. \tag{2.13}$$

Remark 2.2.5 Notice that

$$\mathcal{P}_k^n(x_0) = x_0^k, \tag{2.14}$$

$$\mathcal{P}_k^n(\underline{x}) = c_k(n)\underline{x}^k. \tag{2.15}$$

Therefore the restriction of the polynomials $\mathcal{P}_k^n$ to the real line $x = x_0$ gives the classical real Appell sequence. Furthermore, since $c_k(1) = 1, k = 0, 1, \ldots,$ it follows that

$$\mathcal{P}_k^1(x_0,\underline{x}) = \sum_{s=0}^{k}\binom{k}{s} x_0^{k-s}\underline{x}^s = \sum_{s=0}^{k}\binom{k}{s} x_0^{k-s}(x_1e_1)^s = (x_0 + x_1e_1)^k, \tag{2.16}$$

i.e. for $n = 1$ these polynomials are the complex powers $z^k = (x_0 + x_1e_1)^k$. In addition, since $c_k(n) \neq 1$, or $n \geq 2$, relation (2.15) reflects the difference between the real and complex cases (where the coefficient of $\underline{x}^k$ is 1) and the higher dimensional cases. ♦

We recognize the fact that $\mathcal{P}_k^n$ generalize the complex powers $z^k, z \in \mathbb{C}$ not only with respect to the Appell property and the complex derivative, but also with respect to their structure, as the expressions (2.12) and (2.13) suggest. In fact, in [82] it was proved that these polynomials can be obtained as the Fueter-Sce extension of the complex powers z^{k+n-1} in the case where the underlying Clifford Algebra is generated by an odd number of basis vectors. Moreover, these polynomials can be considered as *special* holomorphic polynomials in the sense of [5]. In [5], a holomorphic polynomial P is said to be special if there exist constants $a_{ij} \in \mathcal{A}_n$ for which

$$P(x) = {\sum_{i,j}}' \bar{x}^i x^j a_{i,j},$$

where the primed sigma stands for a finite sum.

From now on we call the polynomials $\mathcal{P}_k^n$, ***Standard Appell Polynomials*** (SAP).

Remark 2.2.6 Besides the representations given by (2.4), (2.6), (2.11) and (2.12)-(2.13) one can find in [28] a determinant representation of the SAP in terms of x_0 and $\underline{x}$. ♦

Finally, we observe that the representation (2.11) suggests that the equivalence between (2.1) and (2.2)-(2.3) in the classical real case can be extended to the hypercomplex setting. In fact, in [27] the following equivalence was proved.

Theorem 2.2.7 (Theorem 1, [27]) *A holomorphic polynomial sequence* $\left(\mathcal{F}_k\right)_{k\geq 0}$ *is an Appell set if and only if it satisfies the binomial-type identity*

$$\mathcal{F}_k(x) = \mathcal{F}_k(x_0 + \underline{x}) = \sum_{s=0}^{k} \binom{k}{s} \mathcal{F}_{k-s}(\underline{x})\mathcal{F}_s(x_0). \tag{2.17}$$

In the same paper it was also observed that the binomial-type identity (2.17) coincides for holomorphic Appell sets, with the Cauchy-Kovalevskaya extension.

In the following tables we present the first values of the coefficients $T_s^k(n)$ for arbitrary n as well as the first values of $c_k(n)$ for $n = 1, 2, 3, 4$. In addition, we show the first SAP expressed by the use of (2.4), (2.6) and (2.11), for the case $n = 2$. This latter choice has to due with the fact that the SAP $\mathcal{P}_k^2$ will be used in this and in some of the subsequent chapters.

Remark 2.2.8 Among several nice properties of the sequence $\left(c_k(n)\right)_{k\geq 1}$ we would like to highlight the following ones:

1. The coefficients $c_k(2)$ are the generalized central binomial coefficients with weight $\frac{1}{2^k}$, i.e.

$$c_k(2) = \frac{1}{2^k}\binom{k}{\lfloor \frac{k}{2} \rfloor}. \tag{2.18}$$

k	$\mathcal{T}_s^k(n), s=0,1,\ldots,k$					
1	$\frac{n+1}{2n}$	$\frac{n-1}{2n}$				
2	$\frac{n+3}{4n}$	$\frac{n-1}{2n}$	$\frac{n-1}{4n}$			
3	$\frac{(n+5)(n+3)}{8n(n+2)}$	$\frac{3(n-1)(n+3)}{8n(n+2)}$	$\frac{3(n^2-1)}{8n(n+2)}$	$\frac{(n-1)(n+3)}{8n(n+2)}$		
4	$\frac{(n+5)(n+7)}{16n(n+2)}$	$\frac{(n+5)(n-1)}{4n(n+2)}$	$\frac{3(n^2-1)}{8n(n+2)}$	$\frac{n^2-1}{4n(n+2)}$	$\frac{(n+5)(n-1)}{16n(n+2)}$	
5	$\frac{(n+5)(n+7)(n+9)}{32n(n+2)(n+4)}$	$\frac{5(n+5)(n+7)(n-1)}{32n(n+2)(n+4)}$	$\frac{5(n+5)(n^2-1)}{8n(n+2)(n+4)}$	$\frac{5(n+3)(n^2-1)}{16n(n+2)(n+4)}$	$\frac{5(n+5)(n^2-1)}{32n(n+2)(n+4)}$	$\frac{(n+5)(n+7)(n-1)}{32n(n+2)(n+4)}$

Table 2.1: First values of $\mathcal{T}_s^k(n)$ for $k=1,\ldots,5$.

k	$c_k(1)$	$c_k(2)$	$c_k(3)$	$c_k(4)$
1	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$
2	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$
3	1	$\frac{3}{8}$	$\frac{1}{5}$	$\frac{1}{8}$
4	1	$\frac{3}{8}$	$\frac{1}{5}$	$\frac{1}{8}$
5	1	$\frac{5}{16}$	$\frac{1}{7}$	$\frac{5}{64}$

Table 2.2: First values of $c_k(n)$ for $k=1,\ldots,5$ and $n=1,\ldots,4$.

Indeed, from (2.10) we have that $c_k(2)=\frac{k!!}{(k+1)!!}$, for odd k. In such case, by recalling that the relations $(2m-1)!!(2m-2)!!=(2m-1)!$ and $(2m)!!=2^m m!$ are valid for a positive integer m, we conclude that, for odd k,

$$c_k(2)=\frac{k!!(k-1)!!}{2^{\frac{k+1}{2}}\left(\frac{k+1}{2}\right)!(k-1)!}=\frac{k!}{2^{\frac{k+1}{2}}2^{\frac{k-1}{2}}\left(\frac{k+1}{2}\right)!\left(\frac{k-1}{2}\right)!}=\frac{1}{2^k}\binom{k}{\frac{k-1}{2}}.$$

For even k, the relation $c_k(2)=c_{k-1}(2)$, together with last equality give

$$c_k(2)=\frac{1}{2^{k-1}}\binom{k-1}{\frac{k-2}{2}}=\frac{1}{2^k}\binom{k}{\frac{k}{2}}.$$

k	$\mathcal{P}_k^{(2)}(x)$
1	$\frac{3}{4}x + \frac{1}{4}\bar{x}$
2	$\frac{5}{8}x^2 + \frac{1}{4}x\bar{x} + \frac{1}{8}\bar{x}^2$
3	$\frac{35}{64}x^3 + \frac{15}{64}x^2\bar{x} + \frac{9}{64}x\bar{x}^2 + \frac{5}{64}\bar{x}^3$
4	$\frac{63}{128}x^4 + \frac{28}{128}x^3\bar{x} + \frac{18}{128}x^2\bar{x}^2 + \frac{12}{128}x\bar{x}^3 + \frac{1}{128}\bar{x}^4$
5	$\frac{231}{512}x^5 + \frac{105}{512}x^4\bar{x} + \frac{140}{512}x^3\bar{x}^2 + \frac{150}{512}x^2\bar{x}^3 + \frac{105}{512}x\bar{x}^4 + \frac{63}{512}\bar{x}^5$

Table 2.3: First SAP in terms of x and $\bar{x}$.

k	$\mathbf{P}_k(z_1, z_2)$
1	$\frac{1}{2}(z_1e_1 + z_2e_2)$
2	$-\frac{1}{2}(z_1^2 + z_2^2)$
3	$-\frac{3}{8}(z_1^3e_1 + z_1^2 \times z_2e_2 + z_1 \times z_2^2e_1 + z_2^3)$
4	$\frac{3}{8}(z_1^4 + 2z_1^2 + z_2^2 + z_2^4)$
5	$\frac{5}{16}(z_1^5e_1 + z_1^4 \times z_2e_2 + 2z_1^3 \times z_2^2e_1 + 2z_1^2 \times z_2^3e_2 + z_1 \times z_2^4e_1 + z_2^5e_2)$

Table 2.4: First SAP in terms of z_1 and z_2.

It is also worth to underline the similarity of the sequence $c_{2k}(2) = \frac{1}{2^{2k}}\binom{2k}{k}$ and the Catalan numbers

$$\mathcal{C}_k = \frac{1}{k+1}\binom{2k}{k}.$$

Whereas the Catalan numbers are the ratio of the central binomial coefficient $\binom{2k}{k}$ in the $2k$-th row of the Pascal triangle and the total number of binomial coefficients in the k-th row, the $c_{2k}(2)$ are the ratio of the same central binomial coefficient and the sum of all binomial coefficients in the $2k$-th row.

2. Each $c_k(3)$ is given by the reciprocal values of all odd integers, i.e.

$$c_k(3) = \frac{1}{k+2}, \quad k = 1, 3, 5, \ldots.$$

k	$\mathcal{P}_k^{(2)}(x)$
1	$x_0 + \frac{1}{2}\underline{x}$
2	$x_0^2 + x_0\underline{x} + \frac{1}{2}\underline{x}^2$
3	$x_0^3 + \frac{3}{2}x_0^2\underline{x} + \frac{3}{2}x_0\underline{x}^2 + \frac{3}{8}\underline{x}^3$
4	$x_0^4 + 2x_0^3\underline{x} + 3x_0^2\underline{x}^2 + \frac{3}{2}x_0\underline{x}^3 + \frac{3}{8}\underline{x}^4$
5	$x_0^5 + \frac{5}{2}x_0^4\underline{x} + 5x_0^3\underline{x}^2 + \frac{15}{4}x_0^2\underline{x}^3 + \frac{15}{8}x_0\underline{x}^4 + \frac{5}{16}\underline{x}^5$

Table 2.5: First SAP in terms of x_0 and $\underline{x}$.

3. The coefficients $c_k(2)$ and $c_k(4)$ are related by

$$c_k(4) = \frac{2}{k+3}c_k(2), \quad k = 1, 3, 5, \ldots.$$

♦

As it was already referred, the study of polynomials with values in a Clifford Algebra which generalize the positive complex powers z^k has followed many different directions depending on the considered problem. In concrete, the discussion on Appell polynomials with values in a Clifford Algebra revealed to be, by itself, a subject which brought renewed interest to hypercomplex function theory in the last decade.

At the same time, the construction of *rational holomorphic powers*, or simply, of *negative holomorphic powers* has also been subject of investigation in the context of hypercomplex function theory. For instance, the works [8] and [143] focus on the concept of rationality of hyperholomorphic functions and the rational approximation of functions in monogenic Hardy spaces by means of higher order Szegö kernels, respectively.

For our purpose we need the so-called Cauchy kernel in $\mathcal{A}_n$ which may be regarded as the analogue of the Cauchy kernel $\frac{1}{2\pi}\frac{1}{z}$ in $\mathbb{C}$ (see e.g. [23]).

Definition 2.2.9 *For $x \in \mathcal{A}_n\backslash\{0\}$, the* ***Cauchy kernel*** *is given by*

$$\varepsilon_n(x) = \frac{1}{\sigma_n}E_n(x)$$

where

$$E_n(x) = \frac{\bar{x}}{|x|^{n+1}} \tag{2.19}$$

and σ_n is the surface area of the unit ball S^n in $\mathbb{R}^{n+1}$.

The function E_n is a holomorphic paravector-valued, homogeneous function of degree $-n$ which corresponds, for $n = 1$, to the holomorphic function $x^{-1} \cong z^{-1}$ in $\mathbb{C}$.

The following definition extends the concept given in Definition 2.1.1 to *negative Appell powers*. It is based on the complex case where the negative powers $f_k(z) = z^{-k}$ verify $f_k'(z) = -kf_{k+1}(z), z \in \mathbb{C}$. We are aware of the fact that it corresponds to a particular type of holomorphic and homogeneous rational functions with values in $\mathcal{A}_n$, i.e. it is restricted to functions which belong to $\mathcal{H}_k^-$.

Definition 2.2.10 *The sequence* $\left(Q_k\right)_{k\geq 0}$*, where* $Q_k \in \mathcal{H}_k^-$*, is called a* ***generalized Appell sequence*** *with respect to* ∂ *if*

1. $Q_0(x) = \frac{\bar{x}}{|x|^{n+1}}$*, with* $x \neq 0$,
2. $\partial Q_k = -(k+n)Q_{k+1}, k = 1, 2, \ldots.$

In what follows we introduce the Kelvin transform which, applied to an harmonic function, generalizes the inversion on the unit circle in the complex plane to the inversion on the unit sphere in $\mathbb{R}^{n+1}$ (more about its properties and applications in hypercomplex analysis can be found in [76]).

Definition 2.2.11 *Let* $f \in \mathcal{H}_k^+$ *or* $f \in \mathcal{H}_k^-$, $k \in \mathbb{N}_0$, *then*

$$I[f](x) = E_n(x)f(x^{-1}) \tag{2.20}$$

is called the ***Kelvin transform*** *of* f.

We highlight that the Kelvin transform is a bijective map between $\mathcal{H}^+$ and $\mathcal{H}^-$, that is, it maps inner spherical polynomials of degree k into outer spherical functions of degree $-(k+n)$. In the complex setting, we have that

$$I[z^k](z) = z^{-(k+1)} \tag{2.21}$$

$$I[z^{-k}](z) = z^{k-1}, \tag{2.22}$$

where $z \neq 0,\ k \in \mathbb{N}_0$, that is, the Kelvin transform maps positive complex powers to negative ones and vice-versa, with a *shift* of one unit. In particular, the *first* negative power z^{-1} can be obtained by applying the Kelvin transform to $z^0 \equiv 1$. Moreover, since $z^{-1} = E_1(z)$, (2.21) suggests a connection between the Kelvin transform of the k-th complex power and the k-th derivative of $E_1(z)$. The corresponding link for $n \geq 1$ is shown in the following result (see [33]).

Proposition 2.2.12 *Let $\mathcal{P}_k^n$ and E_n be the functions defined by* (2.11) *and* (2.19)*, respectively. Then*

$$E_n^{(k)}(x) = (-1)^k n_{(k)} I[\mathcal{P}_k^n](x). \tag{2.23}$$

Proof. The factorization of E_n in the form

$$\frac{\bar{x}}{|x|^{n+1}} = \frac{\bar{x}\left(|x|^2\right)^{\frac{n-1}{2}}}{\left(|x|^2\right)^{\frac{n+1}{2}}\left(|x|^2\right)^{\frac{n-1}{2}}} = \left(\frac{\bar{x}}{|x|^2}\right)^{\frac{n+1}{2}}\left(\frac{x}{|x|^2}\right)^{\frac{n-1}{2}}$$

allows the use of Leibniz' differentiation rule in order to obtain

$$\begin{aligned}
E_n^{(k)}(x) &= \frac{\partial^k}{\partial x_0^k}\frac{\bar{x}}{|x|^{n+1}} \\
&= \sum_{s=0}^{k}\binom{k}{s}(-1)^k\left(\frac{n+1}{2}\right)_{(k-s)}\left(\frac{n-1}{2}\right)_{(s)}\left(\frac{\bar{x}}{|x|^2}\right)^{\frac{n+1}{2}+k-s}\left(\frac{x}{|x|^2}\right)^{\frac{n-1}{2}+s} \\
&= (-1)^k k!\frac{\bar{x}}{|x|^{n+2k+1}}\sum_{s=0}^{k}\frac{\left(\frac{n+1}{2}\right)_{(k-s)}}{(k-s)!}\frac{\left(\frac{n-1}{2}\right)_{(s)}}{s!}\bar{x}^{k-s}\,x^s \\
&= (-1)^k n_{(k)}\frac{\bar{x}}{|x|^{n+2k+1}}\mathcal{P}_k^n(\bar{x}).
\end{aligned}$$

On the other hand, recalling that $\mathcal{P}_k^n$ have the property of being homogeneous of degree k and applying the Kelvin transform (2.20) we obtain

$$I[\mathcal{P}_k^n](x) = \frac{\bar{x}}{|x|^{n+1}}\mathcal{P}_k^n\left(\frac{\bar{x}}{|x|^2}\right) = \frac{\bar{x}}{|x|^{n+2k+1}}\mathcal{P}_k^n(\bar{x})$$

and the final result follows now at once. ∎

Proposition 2.2.13 *The sequence of outer spherical functions $\left(\mathcal{Q}_k^n\right)_{k\geq 0}$, with*

$$\mathcal{Q}_k^n(x) = \frac{\bar{x}}{|x|^{n+2k+1}}\mathcal{P}_k^n(\bar{x}).$$

forms a generalized Appell sequence.

Proof. Based on its definition, each rational function $\mathcal{Q}_k^n(x)$ is holomorphic, homogeneous of degree $-(k+n)$ and has values in $\mathcal{A}_n$. In addition, since $\mathcal{P}_0^n(x) = 1$ it is clear that $\mathcal{Q}_0(x) = \frac{\bar{x}}{|x|^{n+1}}$, for all $n \geq 1$.

According to (2.4) it is possible to express $\mathcal{Q}_k^n(x)$ as

$$\mathcal{Q}_k^n(x) = \frac{\bar{x}}{|x|^{n+2k+1}}\mathcal{P}_k^n(\bar{x}) = \sum_{s=0}^{k} T_s^k(n)\bar{x}^{k+1-s}x^s|x|^{-(n+2k+1)}$$

and, consequently,

$$\partial\left(\mathcal{Q}_k^n(x)\right) = \partial_0\left(\mathcal{Q}_k^n(x)\right) = \sum_{s=0}^{k} T_s^k(n)\partial_0(\bar{x}^{k+1-s}x^s|x|^{-(n+2k+1)}). \tag{2.24}$$

The computation of $\partial_0(\bar{x}^{k+1-s}x^s|x|^{-(n+2k+1)})$ gives:

$$\begin{aligned}&\partial_0\big(\bar{x}^{k+1-s}x^s|x|^{-(n+2k+1)}\big)\\&= \frac{1}{|x|^{n+3+2k}}\big((k+1-s)\bar{x}^{k-s}x^s|x|^2 + s\bar{x}^{k+1-s}x^{s-1}|x|^2 - (n+1+2k)\bar{x}^{k+1-s}x^s x_0\big).\end{aligned}$$

Writing $x_0 = \frac{x+\bar{x}}{2}$ and $|x|^2 = x\bar{x} = \bar{x}x$ we obtain

$$\begin{aligned}&\partial_0\big(\bar{x}^{k+1-s}x^s|x|^{-(n+2k+1)}\big)\\&= \frac{\bar{x}}{|x|^{n+3+2k}}\Big(-\frac{2s+n-1}{2}\bar{x}^{k-s}x^{s+1} + \frac{2s-2k-n-1}{2}\bar{x}^{k+1-s}x^s\Big),\end{aligned}$$

and, consequently, relation (2.24) can be rewritten as

$$\begin{aligned}\partial_0\left(\mathcal{Q}_k^n(x)\right) = \frac{\bar{x}}{|x|^{n+3+2k}}\sum_{s=0}^{k}\Big(&-\frac{2s+n-1}{2}T_s^k(n)\bar{x}^{k-s}x^{s+1}\\&+\frac{2s-2k-n-1}{2}T_s^k(n)\bar{x}^{k+1-s}x^s\Big),\end{aligned}$$

or

$$\begin{aligned}\partial_0\left(\mathcal{Q}_k^n(x)\right) = \frac{\bar{x}}{|x|^{n+3+2k}}\Big(&-T_0^k(n)\frac{2k+n+1}{2}\bar{x}^{k+1} - T_k^k(n)\frac{2k+n-1}{2}x^{k+1}\\&+\sum_{s=1}^{k}\Big(-T_{s-1}^k(n)\frac{2s+n+3}{2} + T_s^k(n)\frac{2s-2k-n-1}{2}\Big)\bar{x}^{k+1-s}x^s\Big).\end{aligned}$$

Based on the definition (2.5), several relations between the coefficients $T_s^k(n)$ can be derived (see [57] for a list of nice properties of theses numbers). In particular, the following identities can be applied to complete the proof.

$$(2k+n+1)T_0^k(n) = 2(k+n)T_0^{k+1}(n)$$

$$(2k+n-1)T_k^k(n) = 2(k+n)T_{k+1}^{k+1}(n),$$

$$-(2s+n+3)T_{s-1}^k(n) + (2s-2k-n-1)T_s^k(n) = -2(k+n)T_s^{k+1}(n).$$

■

2.3 3D mappings by generalized Joukowski transformations

Due to the role of the complex derivative in the study of conformal mappings in $\mathbb{C}$, it was natural to investigate the role of the hypercomplex derivative from the point of view of quasi-conformal mappings in $\mathbb{R}^{n+1}$ [103, 138]. In fact, conformal mappings in real Euclidean spaces of dimension higher than 2 are restricted to Möbius transformations (Liouville's theorem) which, as was already mentioned, are not holomorphic functions. But obviously, this does not mean that holomorphic functions cannot play an important role in applications to the more general class of quasi-conformal mappings, intensively studied by real and several complex variable methods so far.

The advantage of hypercomplex methods applicable to Euclidean spaces of arbitrary real dimensions (not only of even dimensions like in the case of $\mathbb{C}^n$-methods) is already evident for one of the most important cases in practical applications, i.e. the lowest odd dimensional case of $\mathbb{R}^{2+1}$. Besides other practical reasons, it still allows directly visualization of all geometric mapping properties. In this context it is proved, in [80], that an $\mathcal{A}_2$-holomorphic function with nonvanishing Jacobian determinant locally maps the unit sphere onto ellipsoids and vice versa. Moreover, the relation between the Jacobian determinant and the hypercomplex derivative of a holomorphic function was also studied in [78] and [79]. Finally, we refer to the recent works [114] and [115], in which the coefficient of quasiconformality of the referred 3D mappings is explicitly calculated and the local distortion of M-conformal mappings is studied.

Of course, quasi-conformal 3D-mappings demand more computational capacities than conformal 2D-mappings. But since hypercomplex analysis methods are developed in analogy with complex methods ([16, 42, 46, 78, 79, 80, 109, 114, 115]), the expectations on their efficiency for solving 3D-mapping problems are in general very high. Nevertheless, a systematical work on this subject is still missing, presumably because of missing familiarity with hypercomplex methods and their use in practical problems.

In the 1920's J. L. Walsh (cf. [142]) studied the problem of approximating an arbitrary function of a complex variable by rational functions. Concretely, if a function is known to be continuous (analyticity is not required) on a Jordan curve enclosing the origin then, on this curve, the function can be approximated as closely as desired by a polynomial in z and $\frac{1}{z}$, $R_n(z) = \sum_{k=1}^{n} a_k \left(z^k + \frac{1}{z^k}\right)$ (see also [111]). In fact, restricting to the unit circle we have

$$\sin k\theta = \frac{z^k - z^{-k}}{2i} \qquad \text{and} \qquad \cos k\theta = \frac{z^k + z^{-k}}{2},$$

$\theta \in [0, 2\pi]$, revealing that every trigonometric polynomial is a polynomial in z and $\frac{1}{z}$. A classical example in this category of trigonometric polynomials are the real Chebychev polynomials $T_n = T_n(x)$ with $x \in [-1, 1]$. Their complex parametrization over the unit circle in the form

$$T_k(x) = \cos k\theta = \tfrac{1}{2}(z^k + z^{-k}),$$

with

$$x = \tfrac{1}{2}(z + z^{-1}),$$

already reveals the link between the classical Joukowski transformation (where $k = 1$) and its generalization of higher order in approximation theory.

Later, in the 1980's, the same functions referred above played again an important role in another subject. H. Haruki and M. Barran, with [12] and [83], studied specific functional equations whose unique solution is given by

$$\widetilde{w} = \widetilde{w}(z) = \tfrac{1}{2}(z^k + z^{-k}),$$

where k is a positive integer. The function $\widetilde{w} = w_0 + iw_1$ is said to be a *generalized Joukowski transformation of order k.*

Among other properties it maps the unit circle in the z plane into the interval $[-1, 1]$ of the real axis in the $\widetilde{w}$-plane traced $2k$ times. Moreover, for $k = 1$, i.e. the classical case, symmetric and unsymmetric airfoils are obtained as images of circles with centers sufficiently near to the origin. In the classical *Dictionary of Conformal Representations* [91], for example, or the more recent book *Computational Conformal Mapping* [95], specially dedicated to computational aspects, one can find a lot of details about those symmetric or unsymmetric airfoils.

Following [45] and [46], the generalization to higher dimensions[1] indicates some modification in the use of the standard polar coordinates as well as in the function representation itself. If we use modified polar coordinates in the form

$$z = \rho e^{i(\frac{\pi}{2} - \varphi)} = \rho(\sin\varphi + i\cos\varphi), \qquad \varphi \in [0, 2\pi],$$

then we obtain the interval $[-i, i]$ as the image of the unit circle S^1 under the mapping

$$w = w(z) = \tfrac{1}{2}(z^k - z^{-k}). \tag{2.25}$$

[1]For $k = 1$ see [45] or [46], where this modified treatment of the Joukowski transformation was used for the first time. It allows to connect the 2D case more directly with the corresponding hypercomplex 3D case, where the unit sphere S^2 has a pure vector-valued image in analogy to the pure imaginary image of S^1.

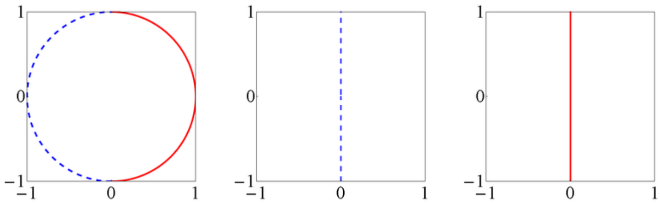

Figure 2.1: Two unit semi-disks and their corresponding images

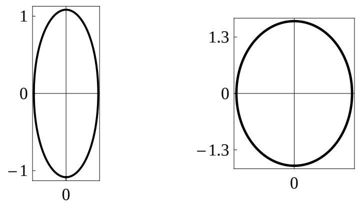

Figure 2.2: The images of disks of radius $\rho = 1.5$ and $\rho = 3$

Moreover the real and imaginary parts of w are obtained in the following form

$$w_0 = \tfrac{1}{2}\big(\rho^k - \tfrac{1}{\rho^k}\big)\cos\big(\tfrac{k}{2}\pi - k\varphi\big), \qquad w_1 = \tfrac{1}{2}\big(\rho^k + \tfrac{1}{\rho^k}\big)\sin\big(\tfrac{k}{2}\pi - k\varphi\big).$$

Thereby circles of radius $\rho \neq 1$ are transformed onto confocal ellipses with semi-axis

$$a = \tfrac{1}{2}\big|\rho^k - \frac{1}{\rho^k}\big|, \qquad b = \tfrac{1}{2}\big(\rho^k + \tfrac{1}{\rho^k}\big)$$

and foci $w = i$ and $w = -i$ while the unit circle is $2k$-fold mapped onto the segment $[-i, i]$.

It should be noted that $b > a$, for all $\rho \neq 1$ and, consequently, the aforementioned ellipses have all the same configurations. Figures 2.1 and 2.2 show the well known images of disks with radii equal or greater than one under the mapping w, for $k = 1$. To stress the double covering of the segment $[-i, i]$, in the case $\rho = 1$, we present the images of both semi-disks separately. The restriction to black and white figures suggested to use also dotted lines.

Figure 2.3 shows two symmetric airfoils that are images of circles centered at points (different from the origin) on the imaginary axis in the complex plane. An unsymmetric Joukowski airfoil (more interesting for studies in aerodynamics) as image of a circle centered at a point in the first quadrant[2] is shown in Figure 2.4.

[2]It is clear that applied to our modified form (2.25) symmetric airfoils are obtained whenever the center is chosen in the imaginary axis, while unsymmetric airfoils are the images of circles with center near to origin passing through $-i$ and i. Of course, these fixpoints are only chosen for some normalization of the mapping and are not essential for its global behavior.

Complex Case - Order $k = 1$

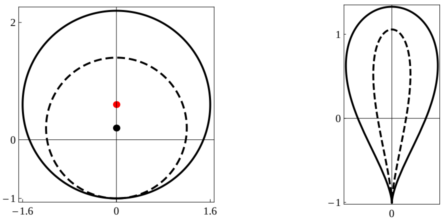

Figure 2.3: Disks with radius $\rho = 1.2$ and $\rho = 1.6$ and center $d = (\rho - 1)i$ and their images

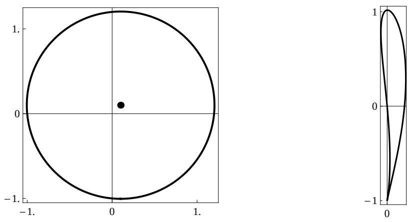

Figure 2.4: Disk with radius $\rho = |1 + d|$ and center $d = 0.1 + 0.1i$ and its image

Complex Case - Order $k = 2$

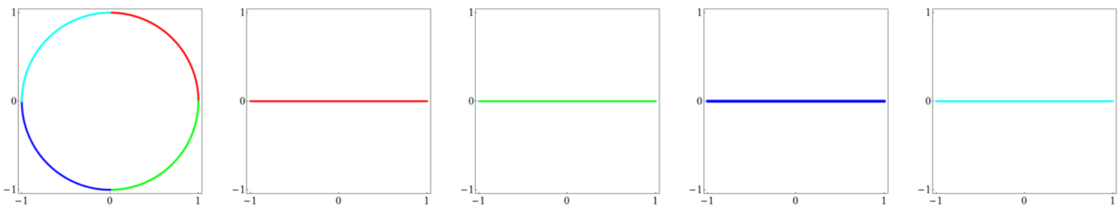

Figure 2.5: Four unit quarter-disks and their corresponding images

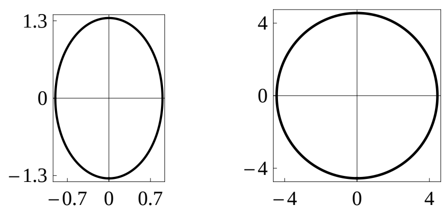

Figure 2.6: The images of semi-disks of radius $\rho = 1.5$ and $\rho = 3$

Complex Case - Order $k = 2$

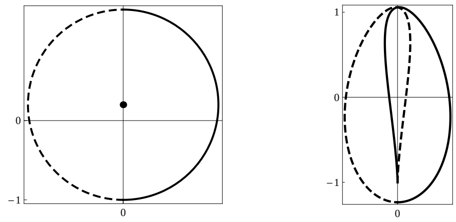

Figure 2.7: Two semi-disks with radius $\rho = 1.2$ and center $d = (\rho - 1)i$ and their images

We illustrate the generalized Joukowski transformation of order 2 in Figures 2.5, 2.6 and 2.7. We underline the fact that, in this case, the mapping function is 4-fold when $\rho = 1$ and 2-fold for $\rho \neq 1$. Due to the higher order of singularities in the origin we should also be aware of more complicated images of circles not centered at the origin (see Figure 2.7).

A suitably higher dimensional analogue of the generalized Joukowski transformation w can be obtained using Clifford algebras. In this case, the chosen holomorphic analogues to z^k and z^{-k} with values in $\mathcal{A}_n \cong \mathbb{R}^{n+1}$ correspond to the SAP $\mathcal{P}_k^n(x)$ and a suitable hypercomplex order derivative of $E_n(x)$.

In [45] and [46] the higher dimensional analogue of the classical Joukowski transform for $n \geq 1$ and $k = 1$ has been studied in detail for the first time. The aim of our work is not only to gain a deeper knowledge in this subject but also to present a second order Joukowski type transformation in $\mathbb{R}^{n+1}$. As such, this section is complemented with some images which were created with *Mathematica*. Part of the work presented in this section has already been published in [33] and [34].

Definition 2.3.1 *Let $x = x_0 + \underline{x} \in \mathcal{A}_n \cong \mathbb{R}^{n+1} \subset \mathcal{C}\ell_{0,n}$. The **generalized hypercomplex Joukowski transformation of order** k, $(k \geq 1)$, is defined as*

$$J_k^n(x) = \alpha_k \left(\mathcal{P}_k^n(x) + \frac{(-1)^k}{n_{(k-1)}} E_n^{(k-1)}(x) \right), \tag{2.26}$$

where α_k is a real constant, $n_{(k-1)}$ is the Pochhammer symbol and $E_n^{(k-1)}(x)$ denotes the hypercomplex derivative of order $(k-1)$ of E_n in (2.19).

Based on relation (2.23) it follows immediately that

$$J_k^n(x) = \alpha_k \Big(\mathcal{P}_k^n(x) + \frac{(-1)^k}{n_{(k-1)}} (-1)^{k-1} n_{(k-1)} I[\mathcal{P}_{k-1}^n](x) \Big) = \alpha_k \Big(\mathcal{P}_k^n(x) - I[\mathcal{P}_{k-1}^n](x) \Big),$$

which means that (2.26) can be expressed only in terms of the well studied SAP as follows.

Proposition 2.3.2 *The generalized hypercomplex Joukowski transformation of order k admits the following representation*

$$J_k^n(x) = \alpha_k\big(\mathcal{P}_k^n(x) - I[\mathcal{P}_{k-1}^n](x)\big) = \alpha_k\Big(\mathcal{P}_k^n(x) - \mathcal{Q}_{k-1}^n(x)\Big).$$

According to (2.16) and (2.21) we have that $J_k^1(x) = \alpha_k(z^k - z^{-k})$. In other words, this means that for $n = 1$ and $\alpha_1 = \frac{1}{2}$ the generalized Joukowski transformation of first order corresponds to the classical Joukowski transformation (2.25). Moreover, formula (2.26) shows a generalization of the hypercomplex Joukowski transformation studied in [45] and [46] which corresponds to consider $n = 2$ and $k = 1$ with $\alpha_1 = \frac{2}{3}$.

In fact, we generalize the different geometrical configurations of the image of spheres with radius $|x| = \rho \geq 1$ given in ([46], Proposition 1) for the 3D case, for arbitrary dimensions.

Proposition 2.3.3 *Let $J_1^n(x)$ be the generalized hypercomplex Joukowski transformation of order 1 given by (2.26) with $\alpha_1 = \frac{n}{n+1}$, then*

1. *The unit sphere S^n is 2-folded mapped onto S^{n-1} (including its interior) in the hyperplane $w_0 = 0$;*
2. *Spheres in $\mathbb{R}^{n+1}$ with radius $1 < \rho < \sqrt[n+1]{\frac{2n}{n-1}}$ are mapped onto $(n+1)-$dimensional oblate spheroids;*
3. *The sphere with radius $\rho = \sqrt[n+1]{\frac{2n}{n-1}}$ is mapped onto the sphere with radius $\frac{\rho}{2}$;*
4. *Spheres in $\mathbb{R}^{n+1}$ with radius $\rho > \sqrt[n+1]{\frac{2n}{n-1}}$ are mapped onto $(n+1)-$dimensional prolate spheroids.*

Proof. In the case $k = 1$

$$J_1^n(x) = \frac{n}{n+1}\left(x_0 + \frac{1}{n}\underline{x} - \frac{\bar{x}}{|x|^{n+1}}\right), \quad x \in \mathcal{A}_n \backslash \{0\}. \tag{2.27}$$

We use here geographic spherical coordinates in $\mathbb{R}^{n+1}$, i.e.

$$\begin{aligned}
x_0 &= \rho\sin(\varphi_0)\\
x_1 &= \rho\cos(\varphi_0)\cos(\varphi_1)\\
x_2 &= \rho\cos(\varphi_0)\sin(\varphi_1)\cos(\varphi_2)\\
&\vdots\\
x_{n-1} &= \rho\cos(\varphi_0)\sin(\varphi_1)\ldots\sin(\varphi_{n-2})\cos(\varphi_{n-1})\\
x_n &= \rho\cos(\varphi_0)\sin(\varphi_1)\ldots\sin(\varphi_{n-2})\sin(\varphi_{n-1})
\end{aligned} \tag{2.28}$$

where $\varphi_0, \varphi_1, \ldots, \varphi_{n-2}$ in $[-\frac{\pi}{2}; \frac{\pi}{2}]$, $\varphi_{n-1} \in [-\pi; \pi]$ and $\rho > 0$. Based on this it is possible to express $J_1^n(x)$ as

$$J_1^n(x) = w_0 + w_1 e_1 + \cdots + w_n e_n,$$

where

$$\begin{aligned} w_0(x) &= \frac{n}{n+1}\Big(1 - \frac{1}{|x|^{n+1}}\Big)x_0 \\ w_i(x) &= \frac{n}{n+1}\Big(\frac{1}{n} + \frac{1}{|x|^{n+1}}\Big)x_i, \quad i = 1, \ldots, n. \end{aligned}$$

Considering the image of the sphere $S_\rho^n = \{x = x_0 + \underline{x} : |x|^2 = \rho\}$ under the mapping J_k^n we observe that since

$$\frac{w_0^2}{\Big[\frac{n}{n+1}\Big(1 - \frac{1}{\rho^{n+1}}\Big)\rho\Big]^2} + \frac{w_1^2}{\Big[\frac{n}{n+1}\Big(\frac{1}{n} + \frac{1}{\rho^{n+1}}\Big)\rho\Big]^2} + \cdots + \frac{w_n^2}{\Big[\frac{n}{n+1}\Big(\frac{1}{n} + \frac{1}{\rho^{n+1}}\Big)\rho\Big]^2} = 1, \tag{2.29}$$

the generalized Joukowski transformation (2.27) maps $(n+1)-$dimensional spheres with radius ρ onto $(n+1)-$dimensional spheroids centered at the origin with polar radii r_1 and equatorial radius r_2, respectively, given by

$$r_1 = \frac{n}{n+1}\left|1 - \frac{1}{\rho^{n+1}}\right|\rho, \qquad r_2 = \frac{n}{n+1}\Big(\frac{1}{n} + \frac{1}{\rho^{n+1}}\Big)\rho. \tag{2.30}$$

When $\rho = 1$ it follows immediately that $r_1 = 0$ and $r_2 = 1$, which means that $J_1^n(S^n)$ gives the $n-$dimensional unit sphere S^{n-1} in the hyperplane $w_0 = 0$. In addition, based on (2.27) we have that $J_1^n(S^n) = \frac{n}{n+1}\big(\frac{1}{n}\underline{x} + \underline{x}\big) = \underline{x}$ and therefore, by means of the referred geographical coordinates one obtains

$$|J_1^n(S^n)| = \cos\varphi_0,$$

that is, the map is $2-$fold, since $\varphi_0 \in [-\frac{\pi}{2}, \frac{\pi}{2}]$.

Based on (2.29) it is easy to see that $r_1 = r_2$ occurs only for $\rho = \sqrt[n+1]{\frac{2n}{n-1}}$, meaning that in this case the spheroid is a sphere of radius $\frac{1}{2}\sqrt[n+1]{\frac{2n}{n-1}} = \frac{\rho}{2}$. The oblate and prolate spheroidal configurations occur when $1 < \rho < \sqrt[n+1]{\frac{2n}{n-1}}$ and $\rho > \sqrt[n+1]{\frac{2n}{n-1}}$, respectively. ■

Figure 2.8 shows, for $n = 2$, the images of both hemispheres with radius equal to one under the mapping J_1^2, while Figure 2.9 shows three spheroids which correspond to the image of spheres with different radii. These pictures reveal the similarity between the complex and the hypercomplex cases, since spheroids of dimension 2 or $n+1$ are obtained as images of spheres with $\rho > 1$, respectively. Nevertheless, and different from the complex case, they also draw attention to the fact that in higher dimensions the configuration of spheroids does not remain the same for all $\rho > 1$ (see Figure 2.2). A last remark concerns the case when $\rho < 1$. In such case, since from (2.30) it follows that $r_1 < r_2$, then the associated spheroids are oblate.

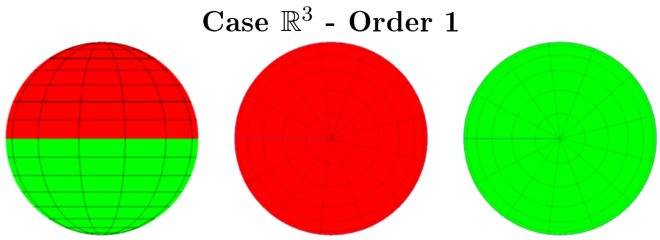

Figure 2.8: Two hemispheres of S^2 and their corresponding images

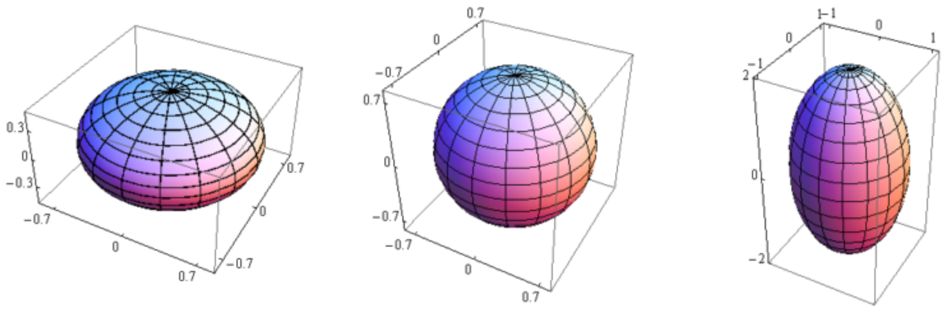

Figure 2.9: The images of spheres of radius $\rho = 1.3$, $\rho = \sqrt[3]{4}$ and $\rho = 3$

After describing some geometric mapping properties of J_1^n in Proposition 2.3.3, we compute now the quasiconformal constant of this function. It is recalled that, in accordance with [115], since J_1^n maps spheres with $\rho > 1$ onto spheroids, it is an M-conformal mapping. Thus, by following the ideas of [46], we consider

$$H_I = \frac{m(E)}{m(B_I)}, \qquad H_O = \frac{m(B_O)}{m(E)}, \tag{2.31}$$

where E denotes the spheroid obtained as image of a sphere with $\rho > 1$ under J_1^n, B_I and B_O denote the inscribed and circumscribed balls of E, respectively, and $m(\cdot)$ stands for the volume. In addition, if

$$K_I(J_1^n) = \sup_{\rho>1} H_I, \qquad K_O(J_1^n) = \sup_{\rho>1} H_O,$$

then, the maximum dilation of J_1^n is defined by $K(J_1^n) = \max\{K_I(J_1^n), K_O(J_1^n)\}$. In such case, if $K(J_1^n) \leq K < \infty$ then J_1^n is K-quasiconformal.

We are now in conditions to state the following:

Proposition 2.3.4 *Let $\rho > 1$ and J_1^n be the generalized Joukowski transformation of first order. Then*

1. *for $1 < \rho < \sqrt[n+1]{\frac{2n}{n-1}}$ the mapping J_1^n is $\frac{1}{n}$-quasiconformal;*

2. *for $\rho > \sqrt[n+1]{\frac{2n}{n-1}}$ the mapping J_1^n is n^n-quasiconformal.*

Proof. We recall here the expression for the volume measure of the n-dimensional ellipsoid

$$\frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} + \cdots + \frac{x_n^2}{a_n^2} = 1,$$

which is given by $V(n) = \beta_n a_1 a_2 \dots a_n$, where $\beta_n = \frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2}+1\right)}$ (i.e. $\beta_n = \frac{2}{n!!}(2\pi)^{\frac{n-1}{2}}$, for even n, while $\beta_n = \frac{1}{(\frac{n}{2})!}\pi^{\frac{n}{2}}$, for odd n).

If we consider $\rho > \sqrt[n+1]{\frac{2n}{n-1}}$ then, according to Proposition 2.3.3, $(n+1)$-dimensional prolate spheroids are obtained as images of spheres with such radii. In this case, according to (2.30) and (2.31), we may consider B_I and B_O to be the balls centered at the origin with radius r_2 and r_1, respectively. Thus we may write

$$H_I = \frac{\beta_n r_1 r_2^n}{\beta_n r_2^{n+1}} = \frac{r_1}{r_2} = n\frac{\rho^{n+1}-1}{\rho^{n+1}+n}$$

$$H_O = \frac{\beta_n r_1^{n+1}}{\beta_n r_1 r_2^n} = \left(\frac{r_1}{r_2}\right)^n = \left[n\frac{\rho^{n+1}-1}{\rho^{n+1}+n}\right]^n,$$

and therefore

$$K_I(J_1^k) = \sup_{\rho > \sqrt[n+1]{\frac{2n}{n-1}}} n\frac{\rho^{n+1}-1}{\rho^{n+1}+n} \qquad \text{and} \qquad K_O(J_1^k) = \sup_{\rho > \sqrt[n+1]{\frac{2n}{n-1}}} \left[n\frac{\rho^{n+1}-1}{\rho^{n+1}+n}\right]^n.$$

Since $K_I(J_1^k)$ and $K_O(J_1^k)$ are increasing functions on the variable ρ then

$$K_I(J_1^k) = \lim_{\rho\to+\infty} n\frac{\rho^{n+1}-1}{\rho^{n+1}+n} \qquad \textit{and} \qquad K_O(J_1^k) = \lim_{\rho\to+\infty} \left[n\frac{\rho^{n+1}-1}{\rho^{n+1}+n}\right]^n,$$

which allows to conclude that $K_I(J_1^k) = n$ and $K_O(J_1^k) = n^n$, and therefore

$$K(J_1^k) = \max\{K_I(J_1^k), K_O(J_1^k)\} = n^n.$$

The proof for the case when $1 < \rho < \sqrt[n+1]{\frac{2n}{n-1}}$ is completely analogous. ■

Remark 2.3.5 For the more general case, when the considered holomorphic function with nonvanishing hypercomplex derivative doesn't map locally spheres onto ellipsoids we may consider, as example, the following definition of quasiconformality given in [138]. ♦

Definition 2.3.6 *Let D and D' be open subsets of $\mathbb{R}^{n+1}$ and $f : D \to D'$ a diffeomorphism. The **inner** and **outer dilations** are defined by*

$$K_I(f) = \sup_{x\in D} H_I(\mathfrak{J}_f(x)), \qquad K_O(f) = \sup_{x\in D} H_O(\mathfrak{J}_f(x)),$$

where

$$H_I(\mathfrak{J}_f(x)) = \frac{|\mathfrak{J}(x,f)|}{m(\mathfrak{J}_f(x))^{m+1}}, \qquad H_O(\mathfrak{J}_f(x)) = \frac{M(\mathfrak{J}_f(x))^{m+1}}{|\mathfrak{J}(x,f)|},$$

$\mathfrak{J}(x,f) = \det \mathfrak{J}_f(x)$ denotes the Jacobian determinant of f, $m(\mathfrak{J}_f(x)) = \min_{|h|=1} |\mathfrak{J}_f(x)h|$ and $M(\mathfrak{J}_f(x)) = \max_{|h|=1} |\mathfrak{J}_f(x)h|$.

The maximum dilation of f is defined by $K(f) = \max\{K_I(f), K_O(f)\}$. If $K(f) \leq K < \infty$ then f is K-quasiconformal.

In what follows we focus on the case $n = 2$, i.e. $\mathbb{R}^3$, and write briefly $\mathcal{P}_k^2(x) = \mathcal{P}_k(x)$, $J_k^2(x) = J_k(x)$ and $c_s(2) = c_s$, i.e.

$$J_k(x) = \alpha_k \left(\mathcal{P}_k(x) - I[\mathcal{P}_{k-1}](x)\right).$$

for $k \geq 1$.

In Proposition 2.3.3 it was proved that the image of S^n under J_1^n is pure vector valued, i.e. $J_1(S^n)$ lies on the hyperplane $w_0 = 0$. Our aim now is to characterize the image of the unit sphere S^2 under J_k^2, for arbitrary k.

Proposition 2.3.7 *The image of the unit sphere S^2 under J_k is pure vector valued, i.e. for $x^* \in S^2$ one has $J_k(x^*) = w_0^k + w_1^k e_1 + w_2^k e_2$, with $w_0^k = 0$.*

Proof. We need to prove that $J_k(x^*)$ is a paravector with vanishing scalar part. By recalling the expression (2.11) of the SAP we have that

$$\begin{aligned} J_k(x^*) &= \alpha_k \sum_{s=0}^{k} c_s \binom{k}{s} x_0^{k-s} \underline{x}^s - \alpha_k (x_0 - \underline{x}) \sum_{s=0}^{k-1} (-1)^s c_s \binom{k-1}{s} x_0^{k-1-s} \underline{x}^s \\ &= \alpha_k (c_k + (-1)^{k-1} c_{k-1}) \underline{x}^k + \alpha_k \sum_{s=1}^{k-1} A(k,s) x_0^{k-s} \underline{x}^s, \end{aligned} \tag{2.32}$$

where

$$A(k,s) = c_s \left[\binom{k}{s} + (-1)^{s-1} \binom{k-1}{s}\right] + (-1)^{s-1} c_{s-1} \binom{k-1}{s-1}.$$

The expression of $A(k,s)$ can be simplified by recalling the special form of the coefficients c_k given in (2.10). Indeed, when s is even, $c_s = c_{s-1}$, and therefore we can write

$$A(k,s) = c_{s-1} \left[\binom{k}{s} - \binom{k-1}{s} - \binom{k-1}{s-1}\right] = 0,$$

while for s odd, based on the fact that $c_k = \frac{k}{k+1} c_{k-2}$ and on well known properties of the binomial coefficients, one obtains

$$\begin{aligned} A(k,s) &= c_s \left[\binom{k}{s} + \binom{k-1}{s} \right] + c_s \frac{s+1}{s} \binom{k-1}{s-1} \\ &= c_s \left[\binom{k}{s} + \frac{k-s}{k} \binom{k}{s} + \frac{s+1}{k} \binom{k}{s} \right] = c_s \binom{k}{s} \frac{2k+1}{k}. \end{aligned}$$

Last relations can be summarized as follows

$$A(k,s) = \begin{cases} c_s \binom{k}{s} \frac{2k+1}{k}, & \text{if } s \text{ is odd} \\ 0, & \text{if } s \text{ is even} \end{cases}.$$

Finally, we observe that

$$c_k + (-1)^{k-1} c_{k-1} = \begin{cases} \frac{2k+1}{k} c_k & \text{if } k \text{ is even} \\ 0 & \text{if } k \text{ is odd} \end{cases},$$

which allows to conclude from (2.32) that the image of the unit sphere for $k > 1$ is given by

$$J_k(x^*) = \alpha_k \frac{2k+1}{k} \underline{x} \sum_{l=0}^{[\frac{k-1}{2}]} c_{2l+1} \binom{k}{2l+1} x_0^{k-(2l+1)} \underline{x}^{2l}. \tag{2.33}$$

Thus, since $\underline{x}^{2l} = (-1)^l |\underline{x}|^{2l}$, we have that $J_k(x^*)$ is pure vector valued and therefore the unit sphere is mapped into the hyperplane $w_0^k = 0$. ■

The last result can also be proved for arbitrary $n \geq 3$, but the corresponding proof relies on more difficult expressions of the $c_k(n)$ and for this reason has been omitted here.

For $k = 1,\ 2,\ 3$ we have the following explicit expressions for the image of $x^* \in S^2$ under the mapping J_k:

$$J_1(x^*) = \alpha_1 \frac{3}{2} \underline{x}, \quad J_2(x^*) = \alpha_2 \frac{5}{2} \underline{x}\, x_0, \quad J_3(x^*) = \alpha_3 \frac{7}{3} \underline{x} \left(\frac{15}{8} x_0^2 - \frac{3}{8} \right).$$

In Proposition 2.3.3 the value of α_1, i.e. $\alpha_1 = \frac{n}{n+1}$, was chosen in such a way that the generalized hypercomplex Joukowski transformation of first order maps the unit sphere S^n in the hyperplane $w_0 = 0$ (in particular S^n is mapped onto S^{n-1}). Similarly, in the 3−dimensional case, the generalized hypercomplex Joukowski transformation of order $k > 1$, with $\alpha_k \neq 0$, guarantees that S^2 is mapped into the hyperplane $w_0^k = 0$ (see Proposition 2.3.7). Indeed, if we write now briefly (2.33) in the form

$$J_k(S^2) = \alpha_k \underline{x} B_k(x)$$

with

$$B_k(x) = B_k(x_0, \underline{x}) = \frac{2k+1}{k} \sum_{l=0}^{\left[\frac{k-1}{2}\right]} c_{2l+1} \binom{k}{2l+1} x_0^{k-(2l+1)} \underline{x}^{2l},$$

we see that the problem of determining α_k for each value of k in the previously mentioned way is solved by

$$\alpha_k = \Big(\max_{|x|^2=1} B_k(x_0, \underline{x}) \Big)^{-1}. \tag{2.34}$$

For $k = 2$ we have that

$$\left| J_2(S^2) \right|^2 = \alpha_2^2 \left(\frac{5}{2} \right)^2 \cos^2\varphi \sin^2\varphi = \alpha_2^2 \left(\frac{5}{4} \right)^2 \sin^2(2\varphi).$$

It is easy to see that in this case $\alpha_2 = \frac{4}{5}$ guarantees the standardization of J_2 in such a way that S^2 is mapped onto S^1. In the same way it is, in principle, possible to determine for every k the corresponding value of α_k in form of (2.34). Since the solution of algebraic equations of higher order becomes involved, it will be obviously more complicated than in those lower dimensional cases.

We proceed now with the case $k = 2$ for which we use the generalized hypercomplex Joukowski transformation with $\alpha_2 = \frac{4}{5}$ as explained before, i.e.

$$\begin{aligned} J_2(x) &= \frac{4}{5} \left(\mathcal{P}_2(x) - I[\mathcal{P}_1](x) \right) = \frac{4}{5} \left(\mathcal{P}_2(x) - \frac{\bar{x}}{|x|^5} \mathcal{P}_1(\bar{x}) \right) \\ &= \frac{4}{5} \left(x_0^2 + x_0 \underline{x} + \frac{1}{2} \underline{x}^2 - \frac{\bar{x}}{|x|^5} \left(x_0 - \frac{1}{2} \underline{x} \right) \right) \end{aligned}$$

Its real components w_i have, in terms of geographic spherical coordinates (see (2.28) for $n = 2$), the following expressions:

$$\begin{aligned} w_0 &= \frac{2}{5} \left(1 - \frac{1}{\rho^5} \right) \rho^2 (-1 + 3\sin^2\varphi), \\ w_1 &= \frac{4}{5} \left(1 + \frac{3}{2\rho^5} \right) \rho^2 \sin\varphi \cos\varphi \cos\theta, \\ w_2 &= \frac{4}{5} \left(1 + \frac{3}{2\rho^5} \right) \rho^2 \sin\varphi \cos\varphi \sin\theta, \end{aligned}$$

As we expected, spheres in $\mathbb{R}^3$ with radius $\rho \neq 1$ are transformed onto spheroids, but this time, we obtain a 2-fold mapping. It is also possible to detect another new property, different from the previous case $k = 1$, namely the effect that the center of the spheroids does not anymore remain on the origin. The shift of the center from the origin occurs in direction of the real w_0-axis and is equal to $\frac{1}{5}\rho^2\left(1-\frac{1}{\rho^5}\right)$. Therefore the polar radius is given by $b=\frac{3}{5}\rho^2\left|1-\frac{1}{\rho^5}\right|$ and the equatorial radius by $a=\frac{1}{5}\rho^2\left(2+\frac{3}{\rho^5}\right)$, so that we have:

$$\frac{\left[w_0-\frac{1}{5}\rho^2\left(1-\frac{1}{\rho^5}\right)\right]^2}{\left[\frac{3}{5}\rho^2\left(1-\frac{1}{\rho^5}\right)\right]^2}+\frac{w_1^2}{\left[\frac{1}{5}\rho^2\left(2+\frac{3}{\rho^5}\right)\right]^2}+\frac{w_2^2}{\left[\frac{1}{5}\rho^2\left(2+\frac{3}{\rho^5}\right)\right]^2}=1.$$

The following result summarizes some properties of the mapping J_2:

Proposition 2.3.8

1. *Spheres with radius* $1<\rho<\sqrt[5]{6}$ *are 2-folded transformed onto oblate spheroids.*
2. *The sphere with radius* $\rho=\sqrt[5]{6}$ *is 2-folded transformed onto the sphere with center* $\left(0,0,\frac{1}{\sqrt[5]{6^3}}\right)$.
3. *Spheres with radius* $\rho>\sqrt[5]{6}$ *are 2-folded transformed onto prolate spheroids.*
4. *The unit sphere* S^2 *is 4-folded mapped onto the unit circle (including its interior) in the hyperplane* $w_0=0$.

Figure 2.10 shows the images of four zones of the unit sphere under the mapping J_2 as consequence of the 4-fold mapping of the unit sphere S^2 to S^1 (cf. with the 2-fold mapping in the case $k=1$ in Figure 2.8). Analogously to $k=1$ where we have shown images of spheres in Figure 2.9, the images in Figure 2.11 are the result of mapping one of the hemispheres with several radius greater than one. Similar to the case $k=1$ the value $\rho=\sqrt[5]{6}$ gives a sphere but now not centered at the origin. Given the importance of the 2−-*dimensional airfoils* obtained via the classical Joukowski transformation in $\mathbb{C}$, it seems natural to ask for their analogues via the generalized hypercomplex Joukowski transform in higher dimensions, in particular in the 3-dimensional case. In this sense, the paper [46] includes images produced with *Maple* of spheres in $\mathbb{R}^3$ centered at points of one of the axes x_1 or x_2 with a small displacement and passing through the endpoints of the unit vectors e_1 and e_2, respectively. In the first image of Figure 2.12 one of these configurations is reproduced, while in the second we show the image of a sphere with $\rho>1$ in a more general position. More concretely, its center is chosen in $(0.1,0.1,0)$ and the point $(-1/2\sqrt{2},-1/2\sqrt{2},0)$ is the corresponding fixpoint of the mapping. Though both cases are images of dislocated from the origin spheres of the same radius $\rho>1$, it seems that the direction of the displacement - only along one axis or not, for example - leads to slightly different images, as the figures suggest. Particularly one can recognize different curvatures of the surfaces. It seems to us not presumptuous to interpret

Case $\mathbb{R}^3$ - Order 2

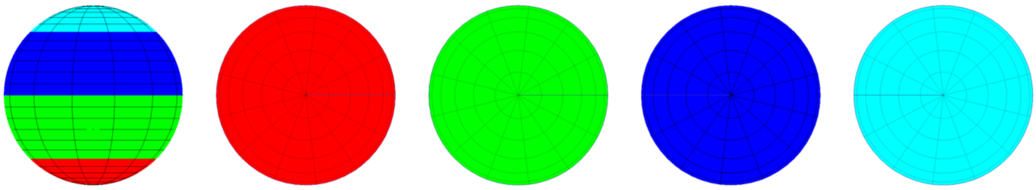

Figure 2.10: The images of the unit sphere

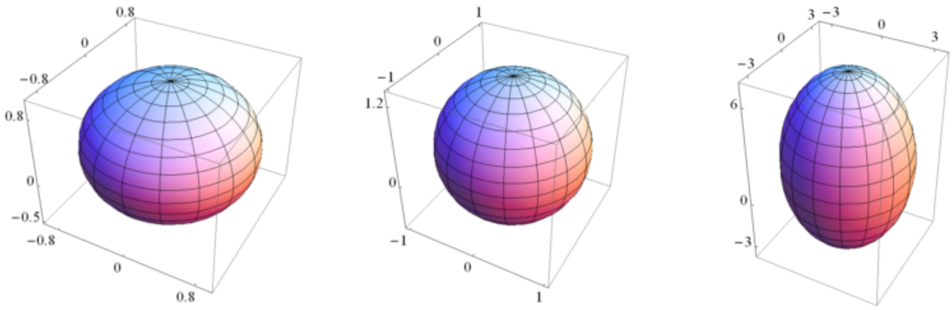

Figure 2.11: The images of hemispheres of radius $\rho = 1.3$, $\rho = \sqrt[5]{6}$ and $\rho = 3$

those figures as some kind of symmetric Joukowski airfoils generalized to 3D and extended in different directions. Figure 2.13 which shows some cuts of the second domain illustrated in Figure 2.12 parallel to the hyperplane $w_1 = w_2$ is in our opinion a very clear illustration of this situation.

If the displacement of the center of the sphere is also done in all three directions unsymmetrically with three different values of the center coordinates, then we get a mapping like the one presented in the Figure 2.14. The cuts which are shown suggest it could be

Hypercomplex Case - Order $k = 1$

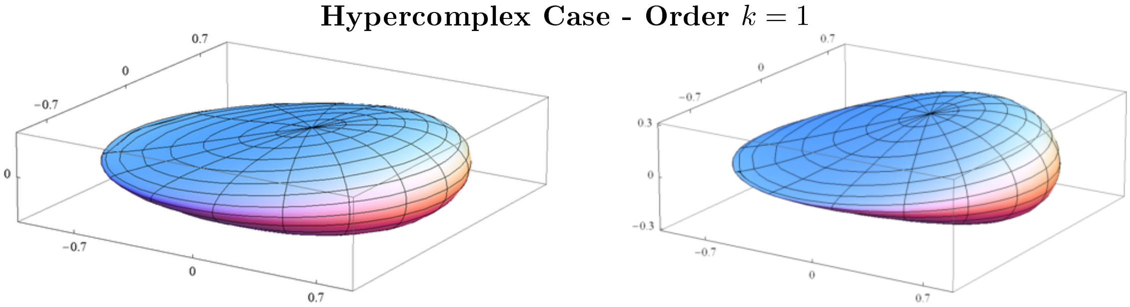

Figure 2.12: The images of spheres of radius $\rho = 1 + |d|$ and centers $d = 0.1e_1$ and $d = 0.1e_1 + 0.1e_2$

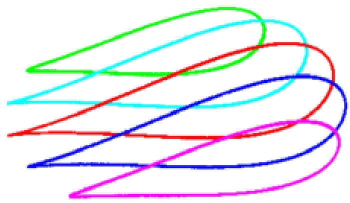

Figure 2.13: Cuts parallel to the hyperplane $w_1 = w_2$

Hypercomplex Case - Order $k = 1$

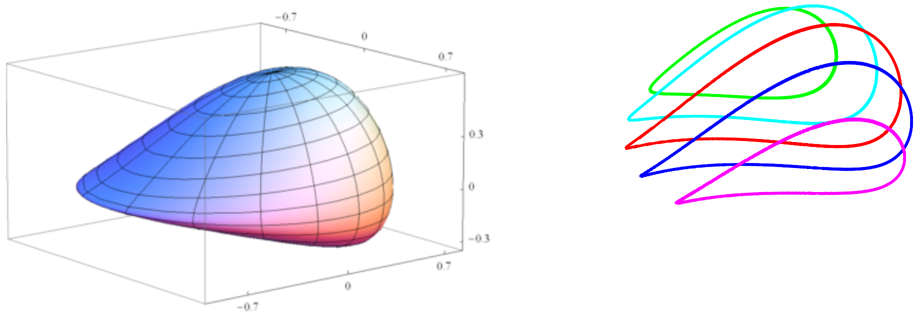

Figure 2.14: The image of a sphere of radius $\rho = 1 + |d|$ and center $d = 0.15e_0 + 0.1e_1 + 0.2e_2$ and some cuts parallel to the hyperplane $w_1 = w_2$

interpreted as some kind of unsymmetric Joukowski airfoil in 3D.

Finally, Figure 2.15 can be interpreted as the 3D analogue of Figure 2.7. Due to the higher order of singularities in the origin we should also be aware of more complicated images of circles and spheres, respectively, with radii different from $\rho = 1$. Nevertheless, we would not exclude the possibility that they could be useful for mathematical models working with more complicated geometric configurations with some singularities, particularly in $\mathbb{R}^3$.

Comments to Chapter 2

The key point of this chapter was the extension of the concept of Appell sequences to holomorphic powers functions of negative degree, by using the Kelvin transform of the standard Appell polynomials. This allowed to consider 3D mappings realized by generalized Joukowski transformation of order k.

Hypercomplex Case - Order $k = 2$

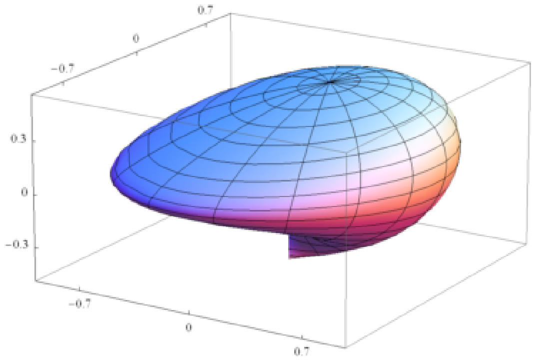

Figure 2.15: The image of a sphere of radius $\rho = 1 + |d|$ and center $d = 0.1e_1 + 0.1e_2$

The presented figures, as well as the associated results seem to support the idea that hypercomplex methods can be a suitable tool for quasi-conformal mappings in $\mathbb{R}^3$. Worth noticing that only basic computational aspects were discussed here. A deeper function theoretic analysis, for instance, the relationship of the hypercomplex derivative and the Jacobian matrix, was not the aim of this work. However, in [45], the reader can find the corresponding results for the hypercomplex case of generalized Joukowski transformations of order $k = 1$.

The last comment concerns the fact that we have produced by holomorphic functions some mappings from the unit sphere in $\mathbb{R}^3$ that remind significant similarities in our opinion with one-wing objects reported in connection with an alternative airframe design, called Blended Wing Body, or BWB (see [1]), which include interesting images of ongoing constructions of one-wing airplanes). There one find the remark that: *The advantages of the BWB approach are efficient high-lift wings and a wide airfoil-shaped body.* With this final remark we leave it as an open question the possibility of application of generalized hypercomplex Joukowski transformations in practical circumstances.

Chapter 3

Appell polynomials and the Pascal simplex

3.1 Introduction

The 1960's and the 1970's were a privileged period for the development of computer aided algebraic manipulation tools. In particular, with the famous *The art of computer programming* by D. Knuth, the technique of combining methods based on *continuous mathematics* together with a controlled manipulation of mathematical formulas in order to solve problems of standard *discrete mathematics*, had gained a renewed interest by many mathematicians and programmers. In many situations this led to new elegant solutions to old problems, but also new combinatorial identities and relations could be detected via the application of this technique. In this connection, we recall two very simple examples which illustrate how to use real and complex valued polynomials in order to obtain combinatorial identities. In concrete, we consider the classic Appell polynomials x^k and z^k, with $x \in \mathbb{R}$ and $z \in \mathbb{C}$ and k a nonnegative integer. In the first case, by considering $x = 2$ and the expansion of $(1+1)^k$ via the binomial theorem, one obtains the well known identity

$$\sum_{s=0}^{k} \binom{k}{s} = \binom{k}{0} + \binom{k}{1} + \cdots + \binom{k}{k} = 2^k,$$

which constitutes one of the huge number of patterns concealed in the Pascal triangle - the sum of all elements of the $k-$th row equals 2^k.

In turn, when considering even powers of $z = 1 + i$, the following relation holds:

$$(1+i)^{2k} = \left(\sqrt{2}e^{i\frac{\pi}{4}}\right)^{2k} = 2^k\Big(\cos\left(\frac{k\pi}{2}\right) + i\sin\left(\frac{k\pi}{2}\right)\Big).$$

As a consequence, by applying the binomial theorem to $(1+i)^{2k}$ and by equating the real and imaginary parts one obtains the binomial identities

$$\sum_{s=0}^{k} \binom{2k}{2s} (-1)^s = 2^k \cos\left(\frac{k\pi}{2}\right) \tag{3.1}$$

$$\sum_{s=0}^{k-1} \binom{2k}{2s+1} (-1)^s = 2^k \sin\left(\frac{k\pi}{2}\right). \tag{3.2}$$

It follows from (3.1) that, when k is odd, the alternating sum of the even terms of the $2k-$th row in the Pascal triangle vanishes. Analogously, the fact that the alternating sum of odd terms in the $2k-$th row vanishes whenever k is even is a direct consequence of (3.2). Although both (3.1) and (3.2) are examples of "hidden" identities of the classical Pascal triangle, these relations can also be observed, if we consider a *Pascal triangle with complex entries*, i.e. a triangular arrangement of complex numbers that gives the coefficients of $x^{k-s}y^s$ in the expansion of $(x+iy)^k$. In this way, the terms which appear in the expansion of $(1+i)^{2k}$ are exactly the same which figure in the $2k-$th row of the aforementioned Pascal triangle with complex entries. Figure 3.1 shows the first even rows of this structure which are labeled as R_{2k}.

R_0				1			
R_2			1	$2i$	-1		
R_4		1	$4i$	-6	$-4i$	1	
R_6	1	$6i$	-15	$-20i$	15	$6i$	-1

Figure 3.1: Pascal triangle with complex entries - first even rows

According to Figure 3.1, identities (3.1) and (3.2) can be now easily verified. In concrete, when adding up all terms of R_{2k} one obtains the real value on the right-hand side of (3.1) or a pure imaginary number whose coefficient is the right-hand side of (3.2), depending whether k is even or odd.

The next example provides a combinatorial identity derived from properties of polynomials with values in the algebra of quaternions. If we consider $q = 1+i+j+k \in \mathbb{H}$ then $q^2 = -2(1-i-j-k) = -2\bar{q}$ and, consequently, $q^3 = -8$. Based on this it follows that

$$(1+i+j+k)^{3k} = (-8)^k,$$

which, by means of the generalized multinomial formula (1.30), allows to deduce the identity

$$\sum_{|\nu|=3k} \binom{3k}{\nu} \vec{q}^{\,\nu} = (-8)^k,$$

or equivalently,

$$\sum_{s_1=0}^{3k} \sum_{s_2=0}^{s_1} \sum_{s_3=0}^{s_2} \binom{3k}{s_1}\binom{s_1}{s_2}\binom{s_2}{s_3} 1^{3k-s_1} \times i^{s_1-s_2} \times j^{s_2-s_3} \times k^{s_3} = (-2)^{3k}. \tag{3.3}$$

In this case, analogously to (3.1) and (3.2), it is possible to construct a 4-dimensional structure which generalizes the Pascal triangle with complex entries. The elements of such structure are the numbers which figure in the expansion (3.3), arranged in a predefined order.

In this context, it seems realistic to think of a detailed study of special Clifford algebra-valued polynomial solutions of generalized Cauchy-Riemann systems in $(n+1)-$dimensional Euclidean spaces as a way to obtain new combinatorial relations. An intuitive approach is related to the fact that, in principle, the properties of holomorphic polynomials in $(n+1)-$real variables significantly depend on the generators $e_1, e_2, \ldots, e_n$ of the underlying 2^n-dimensional Clifford Algebra $\mathcal{C}\ell_{0,n}$ over $\mathbb{R}$. If we add this to the fact that every $\mathcal{A}_n-$valued holomorphic polynomial may be expressed in terms of the generalized powers, then one can think that new combinatorial identities involving expressions with the symmetric product and similar to (3.3), can be deduced.

Of special interest are the four representations of the SAP presented in the previous chapter which involve the coefficients $c_k(n)$ and $T_s^k(n)$. In addition to (2.7), (2.8), (2.9) and (2.10) one can find new properties and combinatorial identities involving these coefficients in [56, 58, 106]. Nevertheless, a complete characterization of the coefficients in the representation (2.6) of the SAP

$$\mathbf{P}_k(z_1, \ldots, z_n) = c_k(n) \sum_{|\nu|=k} z_1^{\nu_1} \times \cdots \times z_n^{\nu_n} \binom{k}{\nu} e_1^{\nu_1} \times \cdots \times e_n^{\nu_n},$$

that is,

$$\binom{k}{\nu} e_1^{\nu_1} \times \cdots \times e_n^{\nu_n} \tag{3.4}$$

was not discussed until now.

The main goal of this chapter is to obtain the explicit expression of these hypercomplex numbers for an arbitrary dimension n. In this way, we generalize the classical *Pascal* $n-$*simplex*, by introducing a Pascal simplex whose entries are the hypercomplex numbers

(3.4). The study of various patterns in such structure and the discussion of properties and combinatorial identities involving the aforementioned coefficients is carried out along this chapter. They reveal great similarities with some of the most classic properties of the Pascal $n-$simplex with real entries. Finally, the last section of this chapter is devoted to the illustration of the special cases of $n = 2$ and $n = 3$, where one can visualize both structures and compare them with the Pascal 2-simplex and the Pascal 3-simplex with real entries.

Our aim here is to contribute for the development of new techniques when working with Clifford algebra-valued holomorphic polynomials, particularly when manipulating expressions involving the symmetric product. A particular important aspect concerns the fact that a complete characterization of the numbers (3.4) will allow, for instance, to obtain new combinatorial identities when comparing the $n = 2$ form of the SAP (2.6) in different representations. This subject will be discussed in Chapter 6. Finally, we refer to the recent works [41, 58, 106, 108] where the reader may find other examples of the potentialities of applying Clifford Analysis methods in combinatorics.

3.2 Pascal $n-$simplex with hypercomplex entries

We start by first recalling the generalized multinomial formula in a commutative or non-commutative ring V. From (1.30), this formula can be written, for $\vec{v} = (v_1, v_2, \dots, v_n) \in V^n$ and $\mu = (\mu_1, \mu_2, \dots, \mu_n) \in \mathbb{N}_0^n$ as

$$\begin{aligned}(v_1 + v_2 + \cdots + v_n)^k &= \sum_{|\mu|=k} \binom{k}{\mu} \vec{v}^{\mu} = \sum_{|\mu|=k} \frac{k!}{\mu_1! \mu_2! \dots \mu_n!} v_1^{\mu_1} \times v_2^{\mu_2} \times \cdots \times v_n^{\mu_n} \\ &= \sum_{s_1=0}^{k} \sum_{s_2=0}^{s_1} \cdots \sum_{s_{n-1}=0}^{s_{n-2}} \mathcal{C}_{(k,s_1,\dots,s_{n-1})} v_1^{k-s_1} \times v_2^{s_1-s_2} \times \cdots \times v_n^{s_{n-1}}, \qquad (3.5)\end{aligned}$$

where

$$\mathcal{C}_{(k,s_1,\dots,s_{n-1})} = \binom{k}{s_1}\binom{s_1}{s_2}\cdots\binom{s_{n-2}}{s_{n-1}}, \qquad (3.6)$$

is a *multinomial coefficient of order* k.

Defining $s_n = 0$, each coefficient $\mathcal{C}_{(k,s_1,\dots,s_{n-1})}$ can be written as

$$\mathcal{C}_{(k,s_1,\dots,s_{n-1})} = \frac{k!}{(k-s_1)!(s_1-s_2)!\dots(s_{n-1}-s_n)!} \qquad (3.7)$$

and this gives the number of ways of splitting a set of k different objects into an ordered sequence of n disjoint subsets, where the number of objects in the 1st subset is $k - s_1$ and in the j^{th} subset is $s_{j-1}-s_j$. It is well known that the multinomial coefficients (3.6) may be arranged/ordered in a n-dimensional structure, usually known as *Pascal hyperpyramid of multinomial coefficients* or *Pascal n-simplex* (see [19] and [144], respectively). In geometry, a n-simplex is a n-dimensional *polytope* which is the convex hull of its $n+1$ vertexes. For example, a 1-simplex is a line segment, a 2-simplex is a triangle and a 3-simplex is a tetrahedron or pyramid. From now on we use the term *simplex* to refer to a regular simplex, i.e. a regular polytope. In general, the convex hull of a subset of size $m+1$ is an m-simplex, called an m-face of the n-simplex. The 0-faces are the *vertexes*, the 1-faces are the *edges* and the $(n-1)$-faces are called the *facets*.

Each Pascal n-simplex is a semi-infinite object which contains n "equal" facets which, in turn, are Pascal $(n-1)$-simplexes. In addition, in each Pascal n-simplex we may consider cross sections parallel to the base where the multinomial coefficients are arranged. For example the k^{th} cross section of the n-simplex contains the numbers $\mathcal{C}_{(k,s_1,\dots,s_{n-1})}$. The Pascal $2-$simplex, usually known as Pascal triangle, has two equal facets/edges which are Pascal $1-$simplexes. Each k^{th} cross section is a line segment, commonly referred as row and here denoted by R_k, which is composed by the coefficients $\mathcal{C}_{(k,s_1)}$. In turn, the Pascal $3-$simplex, in general known by Pascal tetrahedron or Pascal pyramid, has three equal facets which are Pascal triangles. Its k^{th} cross section, usually called k^{th} *layer* and here denoted by $\mathcal{L}_k$, is a triangle which contains the numbers $\mathcal{C}_{(k,s_1,s_2)}$ and whose rows are Pascal 1-simplexes scaled by some integer. Figures 3.2 and 3.3 show the first layers of the Pascal tetrahedron and a representative of its facets, respectively.

In [19] one can find details about the multinomial coefficients and the Pascal n-simplex as well as several identities and references involving these numbers.

Returning our attention back to formula (3.5), we notice that the use of a commutative ring reduces the symmetric product to the ordinary product in V. In particular, if $V=\mathbb{R}$ and $v_1=v_2=\cdots=v_n=1$, the multinomial formula reduces to

$$(1+1+\cdots+1)^k=\sum_{s_1=0}^{k}\sum_{s_2=0}^{s_1}\cdots\sum_{s_{n-1}=0}^{s_{n-2}}\mathcal{C}_{(k,s_1,\dots,s_{n-1})}. \tag{3.8}$$

In this section we intend to study the explicit form of the terms

$$\mathcal{E}_{(k,s_1,\dots,s_{n-1})}=\mathcal{C}_{(k,s_1,\dots,s_{n-1})}e_1^{k-s_1}\times e_2^{s_1-s_1}\times\cdots\times e_n^{s_{n-1}}, \tag{3.9}$$

which appear in the expansion

$$(e_1+\cdots+e_n)^k=\sum_{s_1=0}^{k}\sum_{s_2=0}^{s_1}\cdots\sum_{s_{n-1}=0}^{s_{n-2}}\mathcal{E}_{(k,s_1,\dots,s_{n-1})}.$$

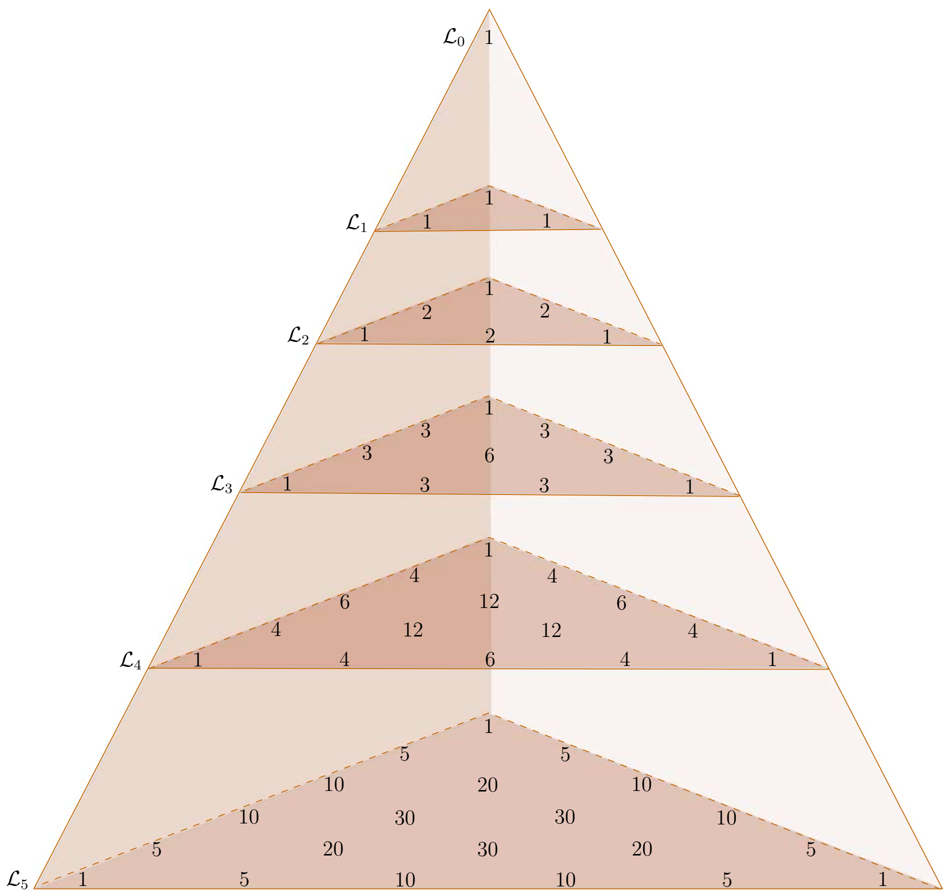

Figure 3.2: Pascal tetrahedron with real entries - $\mathcal{C}_{(k,s_1,s_2)}; k = 0, \ldots, 5$

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}
$$

Figure 3.3: Part of the facet of the Pascal tetrahedron with real entries - $\mathcal{C}_{(k,s_1,0)} = \mathcal{C}_{(k,s_1)}$

By analogy with the real case, we call the hypercomplex numbers $\mathcal{E}_{(k,s_1,\ldots,s_{n-1})}$ *hypercomplex multinomial coefficients of order k*. These numbers are linked with the coefficients (3.4) in the representation of the SAP in the following straightforward way

$$\frac{1}{c_k(n)}\mathbf{P}_k(z_1,\ldots,z_n) = \sum_{s_1=0}^{k}\sum_{s_2=0}^{s_1}\cdots\sum_{s_{n-1}=0}^{s_{n-2}} z_1^{k-s_1}\times z_2^{s_1-s_2}\cdots\times z_n^{s_{n-1}}\mathcal{E}_{(k,s_1,\ldots,s_{n-1})}$$

and consequently, by considering (1.23) it follows immediately from Taylor Theorem, that

$$\mathcal{E}_{(k,s_1,\ldots,s_{n-1})} = \frac{1}{c_k(n)}\frac{1}{(k-s_1)!(s_1-s_2)!\ldots(s_{n-1}-s_n)!}\frac{\partial^k \mathbf{P}_k}{\partial x_1^{k-s_1}\partial x_2^{s_1-s_2}\ldots\partial x_n^{s_{n-1}}}.$$

The next result allows to express $\mathcal{E}_{(k,s_1,\ldots,s_{n-1})}$ explicitly.

Theorem 3.2.1 *The hypercomplex numbers $\mathcal{E}_{(k,s_1,\ldots,s_{n-1})}$ can be written in the form*

$$\mathcal{E}_{(k,s_1,\ldots,s_{n-1})} = (-1)^{\lfloor\frac{k}{2}\rfloor}\prod_{i=1}^{n-1}\binom{\lfloor\frac{s_{i-1}}{2}\rfloor}{\lfloor\frac{s_i}{2}\rfloor}\epsilon_j,$$

where $s_0 = k$ and

$$\epsilon_j = \begin{cases} 1, & \text{if } s_i \text{ is even, } i=0,\ldots,n-1 \\ e_j, & \text{if } s_0,\ldots,s_{j-1} \text{ are odd and } s_j,\ldots,s_{n-1} \text{ are even} \cdot \\ 0, & \text{otherwise} \end{cases}$$

Proof. Consider the pure vector $\lambda = \underline{\lambda} = \lambda_1 e_1 + \cdots + \lambda_n e_n$ where $(\lambda_1,\ldots,\lambda_n)\in\mathbb{R}^n$ and the auxiliary function

$$G_k = G_k(\lambda_1,\ldots,\lambda_n) = \underline{\lambda}^k, \tag{3.10}$$

which can be written, by the use of the generalized multinomial formula (1.30) as

$$\begin{aligned} G_k &= \sum_{s_1=0}^{k}\sum_{s_2=0}^{s_1}\cdots\sum_{s_{n-1}=0}^{s_{n-2}} \lambda_1^{k-s_1}\lambda_2^{s_1-s_2}\ldots\lambda_n^{s_{n-1}}\mathcal{C}_{(k,s_1,\ldots,s_{n-1})}e_1^{k-s_1}\times e_2^{s_1-s_2}\times\cdots\times e_n^{s_n} \\ &= \sum_{s_1=0}^{k}\sum_{s_2=0}^{s_1}\cdots\sum_{s_{n-1}=0}^{s_{n-2}} \lambda_1^{k-s_1}\lambda_2^{s_1-s_2}\ldots\lambda_n^{s_{n-1}}\mathcal{E}_{(k,s_1,\ldots,s_{n-1})}. \end{aligned}$$

Observe that $\mathcal{E}_{(k-s_1,s_1-s_2,\ldots,s_{n-1})}$ are the Taylor coefficients of the function $G_k(\lambda_1,\ldots,\lambda_n)$ and therefore can be calculated as

$$\mathcal{E}_{(k-s_1,s_1-s_2,\ldots,s_{n-1})} = \prod_{i=1}^{n}\frac{1}{(s_{i-1}-s_i)!}\frac{\partial^k G_k(0,\ldots,0)}{\partial\lambda_1^{k-s_1}\partial\lambda_2^{s_1-s_2}\ldots\partial\lambda_n^{s_{n-1}-s_n}}, \tag{3.11}$$

with $s_0 = k$ and $s_n = 0$.

On the other hand, since $G_1(\lambda_1,\ldots,\lambda_n)$ is a pure vector and

$$G_k(\lambda_1,\ldots,\lambda_n) = \big(G_1(\lambda_1,\ldots,\lambda_n)\big)^k,$$

it is clear that, depending on the parity of k, the function $G_k(\lambda_1,\ldots,\lambda_n)$ is a scalar or a pure vector. More precisely,

$$G_k(\lambda_1,\ldots,\lambda_n) = \begin{cases} (-1)^{\frac{k}{2}}(\lambda_1^2+\cdots+\lambda_n^2)^{\frac{k}{2}}, & \text{if } k \text{ is even} \\ (-1)^{\frac{k-1}{2}}(\lambda_1^2+\cdots+\lambda_n^2)^{\frac{k-1}{2}}(\lambda_1 e_1+\cdots+\lambda_n e_n), & \text{if } k \text{ is odd} \end{cases}. \tag{3.12}$$

The proof continues by considering the parity of k.

I. *k* even:

By the use of the classical multinomial formula, (3.12) reads as

$$G_k = (-1)^{\frac{k}{2}} \sum_{r_1=0}^{\frac{k}{2}} \sum_{r_2=0}^{r_1} \cdots \sum_{r_{n-1}=0}^{r_{n-2}} \binom{r_0}{r_1}\binom{r_1}{r_2}\cdots\binom{r_{n-2}}{r_{n-1}} \lambda_1^{2r_0-2r_1}\lambda_2^{2r_1-2r_2}\ldots\lambda_n^{2r_{n-1}-2r_n},$$

where $r_0 = \frac{k}{2}$ and $r_n = 0$. Therefore

$$\begin{aligned} \frac{\partial^k G_k(0,\ldots,0)}{\partial\lambda_1^{k-2r_1}\ldots\partial\lambda_n^{2r_{n-1}-2r_n}} &= (-1)^{\frac{k}{2}}\binom{r_0}{r_1}\binom{r_1}{r_2}\cdots\binom{r_{n-2}}{r_{n-1}}(2r_0-2r_1)!\ldots(2r_{n-1}-2r_n)! \\ &= (-1)^{\frac{k}{2}}\prod_{i=1}^{n}\binom{r_{i-1}}{r_i}(2r_{i-1}-2r_i)! \end{aligned}$$

and all the other partial derivatives of order k of G_k vanish. This means that (3.11) simplifies to

$$\mathcal{E}_{(k,s_1,\ldots,s_{n-1})} = \begin{cases} (-1)^{\frac{k}{2}}\displaystyle\prod_{i=1}^{n-1}\binom{r_{i-1}}{r_i}, & \text{if } s_i = 2r_i, i=1,\ldots,n \\ 0, & \text{otherwise} \end{cases},$$

which completes the proof of this case.

II. *k* odd:

In such case, the expression of G_k has the form

$$G_k = (-1)^{\frac{k-1}{2}} \sum_{r_1=0}^{\frac{k-1}{2}} \sum_{r_2=0}^{r_1} \cdots \sum_{r_{n-1}}^{r_{n-2}} \binom{\frac{k-1}{2}}{r_1}\binom{r_1}{r_2}\cdots\binom{r_{n-2}}{r_{n-1}} \lambda_1^{k-1-2r_1}\lambda_2^{2r_1-2r_2}\ldots\lambda_n^{2r_{n-1}}\underline{\lambda},$$

which is equivalent to

$$\begin{aligned} G_k = (-1)^{\frac{k-1}{2}} \sum_{r_1=0}^{\frac{k-1}{2}} \sum_{r_2=0}^{r_1} \cdots \sum_{r_{n-1}}^{r_{n-2}} \binom{\frac{k-1}{2}}{r_1}\binom{r_1}{r_2}\cdots\binom{r_{n-2}}{r_{n-1}} \Big(\lambda_1^{k-2r_1}\lambda_2^{2r_1-2r_2}\ldots\lambda_n^{2r_{n-1}}e_1 \\ + \lambda_1^{k-1-2r_1}\lambda_2^{2r_1+1-2r_2}\ldots\lambda_n^{2r_{n-1}}e_2+\cdots+\lambda_1^{k-1-2r_1}\lambda_2^{2r_1-2r_2}\ldots\lambda_n^{2r_{n-1}+1}e_n\Big). \end{aligned}$$

We can now argue that the partial derivatives of order k of G_k do not vanish only in one of the n cases listed below. For each case we also list the corresponding expression of the hypercomplex multinomial coefficients.

[1.] $s_i = 2r_i$, for $i = 1, \ldots, n-1$

$$\mathcal{E}_{(k,s_1,\ldots,s_{n-1})} = (-1)^{\frac{k-1}{2}} \binom{\frac{k-1}{2}}{\frac{s_1}{2}} \binom{\frac{s_1}{2}}{\frac{s_2}{2}} \cdots \binom{\frac{s_{n-2}}{2}}{\frac{s_{n-1}}{2}} e_1;$$

[2.] $s_1 = 2r_1 + 1$ and $s_i = 2r_i$, for $i = 2, \ldots, n-1$

$$\mathcal{E}_{(k,s_1,\ldots,s_{n-1})} = (-1)^{\frac{k-1}{2}} \binom{\frac{k-1}{2}}{\frac{s_1-1}{2}} \binom{\frac{s_1-1}{2}}{\frac{s_2}{2}} \cdots \binom{\frac{s_{n-2}}{2}}{\frac{s_{n-1}}{2}} e_2;$$

$\vdots$

[j.] $s_i = 2r_i + 1$, for $i = 1, \ldots, j-1$ and $s_i = 2r_i$, for $i = j, \ldots, n-1$

$$\mathcal{E}_{(k,s_1,\ldots,s_{n-1})} = (-1)^{\frac{k-1}{2}} \binom{\frac{k-1}{2}}{\frac{s_1-1}{2}} \cdots \binom{\frac{s_{j-1}-1}{2}}{\frac{s_j}{2}} \binom{\frac{s_j}{2}}{\frac{s_{j+1}}{2}} \cdots \binom{\frac{s_{n-2}}{2}}{\frac{s_{n-1}}{2}} e_j;$$

$\vdots$

[n.] $s_i = 2r_i + 1$, for $i = 1, \ldots, n-1$

$$\mathcal{E}_{(k,s_1,\ldots,s_{n-1})} = (-1)^{\frac{k-1}{2}} \binom{\frac{k-1}{2}}{\frac{s_1-1}{2}} \binom{\frac{s_1-1}{2}}{\frac{s_2-1}{2}} \cdots \binom{\frac{s_{n-2}-1}{2}}{\frac{s_{n-1}-1}{2}} e_n.$$

This completes the proof. ■

Remark 3.2.2 According to this last result, the coefficients $\mathcal{E}_{(k,s_1,\ldots,s_{n-1})}$ vanish whenever the $n-$tuple $(k, s_1, s_2, \ldots, s_{n-1})$ has one of the following forms:

i. k is even and s_i is odd for at least one i;

ii. k is odd and there exists $j \geq i$ such that s_j is odd and s_i is even.

♦

Based on Theorem 3.2.1 it follows immediately that when k is even the non-vanishing terms of the hypercomplex Pascal $n-$simplex are associated with real multinomial coefficients of order $\frac{k}{2}$. In fact, by considering in (3.10) the case when $\lambda_1 = \lambda_2 = \cdots = \lambda_n = 1$, one obtains the following result which can be interpreted as an hypercomplex version of identity (3.8).

Corollary 3.2.3

$$\sum_{s_1=0}^{k}\sum_{s_2=0}^{s_1}\cdots\sum_{s_{n-1}=0}^{s_{n-2}}\mathcal{E}_\nu=\begin{cases}(-n)^{\frac{k}{2}}, & \text{if } k \text{ is even}\\ (-n)^{\frac{k-1}{2}}(e_1+e_2+\cdots+e_n), & \text{if } k \text{ is odd}\end{cases}.$$

The following identity can now be established easily.

Corollary 3.2.4 *If $\mu,\xi\in\mathbb{N}_0^n$ with $\mu=(\mu_1,\ldots,\mu_n)$ and $\xi=(\xi_1,\ldots,\xi_n)$, then*

$$\sum_{|\mu|=4l}\binom{4l}{\mu}e_1^{\mu_1}\times\cdots\times e_n^{\mu_n}=\sum_{|\xi|=2l}\binom{2l}{\xi}=n^{2l},\ l\in\mathbb{N}_0.$$

Proof. By considering the special case of $k=4l$ in the previous result and the case of $k=2l$ in (3.8) it follows immediately that

$$(e_1+\cdots+e_n)^{4l}=\sum_{s_1=0}^{4l}\sum_{s_2=0}^{s_1}\cdots\sum_{s_{n-1}=0}^{s_{n-2}}\mathcal{E}_{(4l,s_1,\ldots,s_{n-1})}=n^{2l}=\sum_{|\xi|=2l}\binom{2l}{\xi}.$$

■

Finally we present a recurrence property concerning $\mathcal{E}_{(k,s_1,\ldots,s_{n-1})}$.

Proposition 3.2.5 *The hypercomplex numbers $\mathcal{E}_{(k,s_1,\ldots,s_{n-1})}$ verify the recurrence relation*

$$\mathcal{E}_{(k,s_1,\ldots,s_{n-1})}=\mathcal{E}_{(k-1,s_1,\ldots,s_{n-1})}e_1+\mathcal{E}_{(k-1,s_1-1,\ldots,s_{n-1})}e_2+\cdots+\mathcal{E}_{(k-1,s_1-1,\ldots,s_{n-1}-1)}e_n \quad (3.13)$$

with initial conditions $\mathcal{E}_{(0,0,\ldots,0)}=1$ and $\mathcal{E}_{(s_0,s_1,\ldots,s_{n-1})}=0$, for $s_i>s_{i-1}$.

Proof. Based on the general recursion formula (1.15) we can express (3.8) as

$$\mathcal{E}_{(k,s_1,s_2,\ldots,s_{n-1})}=\frac{1}{k}\mathcal{C}_{(k,s_1,s_2,\ldots,s_{n-1})}\Big[(k-s_1)e_1^{k-1-s_1}\times e_2^{s_1-s_2}\times\cdots\times e_n^{s_{n-1}}e_1$$
$$+(s_1-s_2)e_1^{k-s_1}\times e_2^{s_1-1-s_2}\times\cdots\times e_n^{s_{n-1}}e_2+\cdots+s_{n-1}e_1^{k-s_1}\times e_2^{s_1-s_2}\times\cdots\times e_n^{s_{n-1}-1}e_n\Big].$$

In addition, it follows straightforward from (3.7) that

$$\frac{k-s_1}{k}\mathcal{C}_{(k,s_1,s_2,\ldots,s_{n-1})}=\mathcal{C}_{(k-1,s_1,s_2,\ldots,s_{n-1})}$$

$$\frac{s_1-s_2}{k}\mathcal{C}_{(k,s_1,s_2,\ldots,s_{n-1})}=\mathcal{C}_{(k-1,s_1-1,s_2,\ldots,s_{n-1})}$$

and, in general,

$$\frac{s_i-s_{i+1}}{k}\mathcal{C}_{(k,s_1,s_2,\ldots,s_{n-1})}=\mathcal{C}_{(k-1,s_1-1,s_2-1,\ldots,s_i-1,s_{i+1},s_{i+2},\ldots,s_{n-1})},$$

where, as usual, we treat any empty entry of $\mathcal{C}_{(s_0,s_1,s_2,\dots,s_{n-1})}$ as zero. This completes the proof. ■

From the aforementioned properties, it is clear that the numbers $\mathcal{E}_{(k,s_1,\dots,s_{n-1})}$ can be arranged in a $n-$dimensional structure analogue to the Pascal n-simplex, which we designate by *Pascal $n-$simplex with hypercomplex entries* or *hypercomplex Pascal $n-$simplex.* From now on, we use the same terminology used for the Pascal $n-$simplex with real entries whenever referring to the Pascal $n-$simplex with hypercomplex entries, in particular the terms *cross section* and *facet.* This n-dimensional Pascal simplex possesses $(n-1)$-dimensional facets which look like the Pascal simplex of one dimension less. The inner structure of the Pascal n-simplex reveals also similarities with the real case where one has (cf. [19])

$$\mathcal{C}_{(k,s_1,\dots,s_{n-1})} = \mathcal{C}_{(k-1,s_1,\dots,s_{n-1})} + \mathcal{C}_{(k-1,s_1-1,\dots,s_{n-1})} + \cdots + \mathcal{C}_{(k-1,s_1-1,\dots,s_{n-1}-1)}, \tag{3.14}$$

i.e. each "interior" number is just the sum of n numbers in the previous cross section. Indeed, the recurrence relation (3.13) can be seen as the hypercomplex version of (3.14), where the generators $e_1,\dots,e_n$ come naturally.

3.3 Special cases

Pascal triangle with hypercomplex entries

We concentrate now on the hypercomplex numbers given by (3.9) for the case where $n=2$, i.e. $\mathcal{E}_{(k,s)}$ for $k \geq 0$ and $s = 0,1,\dots,k$. According to the previous section these numbers may be disposed in a Pascal triangle with hypercomplex entries, whose first five ordered rows R_i are displayed in Figure 3.4.

According to Theorem 3.2.1, the explicit expression of the hypercomplex numbers $\mathcal{E}_{(k,s)}$ reads as follows.

I. If k is even then

$$\mathcal{E}_{(k,s)} = \begin{cases} (-1)^{\frac{k}{2}} \binom{\frac{k}{2}}{\frac{s}{2}}, & \text{if } s \text{ even;} \\ 0, & \text{if } s \text{ odd.} \end{cases}$$

II. If k is odd then

$$\mathcal{E}_{(k,s)} = \begin{cases} (-1)^{\frac{k-1}{2}} \binom{\frac{k-1}{2}}{\frac{s}{2}} e_1, & \text{if } s \text{ even;} \\ (-1)^{\frac{k-1}{2}} \binom{\frac{k-1}{2}}{\frac{s-1}{2}} e_2, & \text{if } s \text{ odd.} \end{cases}$$

$$
\begin{array}{lccccccccc}
R_0 & & & & & \binom{0}{0}e_1^0 \times e_2^0 & & & & \\
R_1 & & & & \binom{1}{0}e_1^1 & & \binom{1}{1}e_2^1 & & & \\
R_2 & & & \binom{2}{0}e_1^2 & & \binom{2}{1}e_1^1 \times e_2^1 & & \binom{2}{2}e_2^2 & & \\
R_3 & & \binom{3}{0}e_1^3 & & \binom{3}{1}e_1^2 \times e_2^1 & & \binom{3}{2}e_1^1 \times e_2^2 & & \binom{3}{3}e_2^3 & \\
R_4 & \binom{4}{0}e_1^4 & & \binom{4}{1}e_1^3 \times e_2^1 & & \binom{4}{2}e_1^2 \times e_2^2 & & \binom{4}{3}e_1^1 \times e_2^3 & & \binom{4}{4}e_2^4
\end{array}
$$

Figure 3.4: Pascal triangle with hypercomplex entries - first rows

Based on this, the Pascal triangle presented in Figure 3.4 can be obtained at once (see Figure 3.5).

$$
\begin{array}{lccccccccc}
R_0 & & & & & 1 & & & & \\
R_1 & & & & e_1 & & e_2 & & & \\
R_2 & & & -1 & & 0 & & -1 & & \\
R_3 & & -e_1 & & -e_2 & & -e_1 & & -e_2 & \\
R_4 & 1 & & 0 & & 2 & & 0 & & 1
\end{array}
$$

Figure 3.5: Pascal triangle with hypercomplex entries - $\mathcal{E}_{(k,s)}; k = 0, \ldots, 4$

Another interesting relation concerns the sum of all the elements of a specific row. In concrete, the $n = 2$ form of Corollary 3.2.3 states that

$$
\sum_{s=0}^{k} \mathcal{E}_{(k,s)} = \begin{cases} (-2)^{\frac{k}{2}}, & \text{if k is even} \\ (-2)^{\frac{k-1}{2}}(e_1 + e_2), & \text{if k is odd} \end{cases}
$$

In Figure 3.5 it is visible that the sum of all elements of, say R_4, results in $4 = (-2)^{\frac{4}{2}}$, which is in accordance with the previous result. In addition, by recalling Corollary 3.2.4 we can establish the following identity between $\mathcal{E}_{(4,s)}$ and $\mathcal{C}_{(2,s)}$ (cf. Figures 3.3 and 3.5)

$$
\sum_{s=0}^{4} \binom{4}{s} e_1^{4-s} \times e_2^s = \sum_{s=0}^{2} \binom{2}{s} = 2^2 = 4,
$$

or equivalently,

$$\sum_{s=0}^{4} \mathcal{E}_{(4,s)} = \sum_{s=0}^{2} \mathcal{C}_{(2,s)} = 2^2 = 4.$$

It is also visible in Figure 3.5 the fact that the values of the rows change over alternately between real and pure vector values. Despite their different nature, according to Proposition 3.2.5 any two consecutive rows are related by the following equality

$$\mathcal{E}_{(k,s)} = \mathcal{E}_{(k-1,s)} e_1 + \mathcal{E}_{(k-1,s-1)} e_2, \tag{3.15}$$

where $s = 1, \ldots, k-1$. Formula (3.15) reflects the similarity with the real case where

$$\mathcal{C}_{(k,s)} = \binom{k}{s} = \binom{k-1}{s} + \binom{k-1}{s-1},$$

with $s = 1, \ldots, k-1$. Figure 3.6 exemplifies this recurrence relation for the first rows of the Pascal triangle with hypercomplex entries. For instance, the second element of R_3, i.e. $-e_2$, is obtained by multiplying the first and second elements of R_2 by e_2 and e_1, respectively.

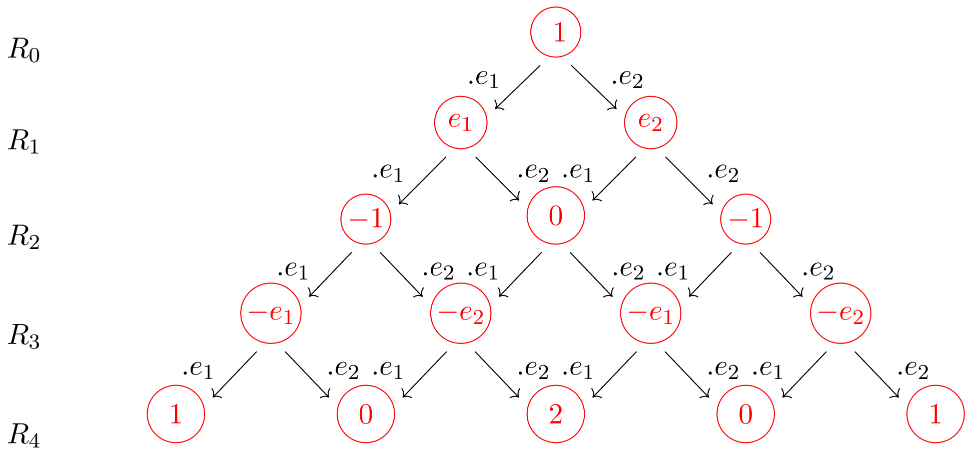

Figure 3.6: Relation between two consecutive rows in the hypercomplex Pascal triangle

Remark 3.3.1 A number of authors have constructed arithmetic triangles by choosing as their elements the numbers which satisfy a recurrence relation of the form

$$\mathcal{F}_{(k,s)} = \mathcal{F}_{(k-1,s)} A(k,s) + \mathcal{F}_{(k-1,s-1)} B(k,s),$$

with appropriate coefficients $A(k,s)$ and $B(k,s)$ and initial conditions. According to (3.15), the hypercomplex Pascal triangle corresponds to the particular choice of $A(k,s) = e_1$ and $B(k,s) = e_2$. ♦

Pascal tetrahedron with hypercomplex entries

We focus now on the hypercomplex numbers given by (3.9) for the case where $n = 3$, i.e. $\mathcal{E}_{(k,s_1,s_2)}$ for $k \geq 0$, $s_1 = 0, 1, \ldots, k$ and $s_2 = 0, 1, \ldots, s_1$. As was already mentioned, these numbers can be arranged in a tetrahedron which is bounded by three triangles which, in turn, look like the Pascal triangle with hypercomplex entries. In what follows each k^{th} cross section or layer $\mathcal{L}_k$ is written in ordered rows[1] as arrangement of the different monomials corresponding to the increasing row-index s_2. For instance, the layer $\mathcal{L}_3$ has the form

$s_1 = 0$		$\binom{3}{0}e_1^3$		
$s_1 = 1$	$\binom{3}{1}\binom{1}{0}e_1^2 \times e_2$		$\binom{3}{1}\binom{1}{1}e_1^2 \times e_3$	
$s_1 = 2$	$\binom{3}{2}\binom{2}{0}e_1 \times e_2^2$	$\binom{3}{2}\binom{2}{1}e_1 \times e_2 \times e_3$	$\binom{3}{2}\binom{2}{2}e_1 \times e_3^2$	
$s_1 = 3$	$\binom{3}{3}\binom{3}{0}e_2^3$	$\binom{3}{3}\binom{3}{1}e_2^2 \times e_3$	$\binom{3}{3}\binom{3}{2}e_2 \times e_3^2$	$\binom{3}{3}\binom{3}{0}e_3^3$

Figure 3.7: Layer $\mathcal{L}_3$ of the Pascal tetrahedron with hypercomplex entries

As an example we consider the central element of $\mathcal{L}_3$:

$$\binom{3}{2}\binom{2}{1}e_1 \times e_2 \times e_3 = \binom{3}{2}\binom{2}{1}\frac{1}{3!}\big(e_1(e_2e_3 + e_3e_2) + e_2(e_1e_3 + e_3e_1) + e_3(e_1e_2 + e_2e_1)\big) = 0.$$

The expression of the coefficients $\mathcal{E}_{(k,s_1,s_2)}$ which appear in the expansion of $(e_1 + e_2 + e_3)^k$, can be obtained by the use of Theorem 3.2.1, i.e.

I. If k is even then

$$\mathcal{E}_{(k,s_1,s_2)} = \begin{cases} (-1)^{\frac{k}{2}}\begin{pmatrix}\frac{k}{2}\\ \frac{s_1}{2}\end{pmatrix}\begin{pmatrix}\frac{s_1}{2}\\ \frac{s_2}{2}\end{pmatrix}, & \text{if } s_1, s_2 \text{ even}\\ 0, & \text{otherwise}\end{cases}$$

[1]The index s_1 is increasing in N-S direction and consequently $k - s_1$ is decreasing in the same direction, whereas s_2 increases in W-E direction. Since $\binom{k}{s_1}\binom{s_1}{s_2} = \binom{k}{s_1}\binom{s_1}{s_1-s_2} = \binom{k}{k-s_1}\binom{s_1}{s_2}$ all elements with the same corresponding index in these three directions are the same (symmetry property).

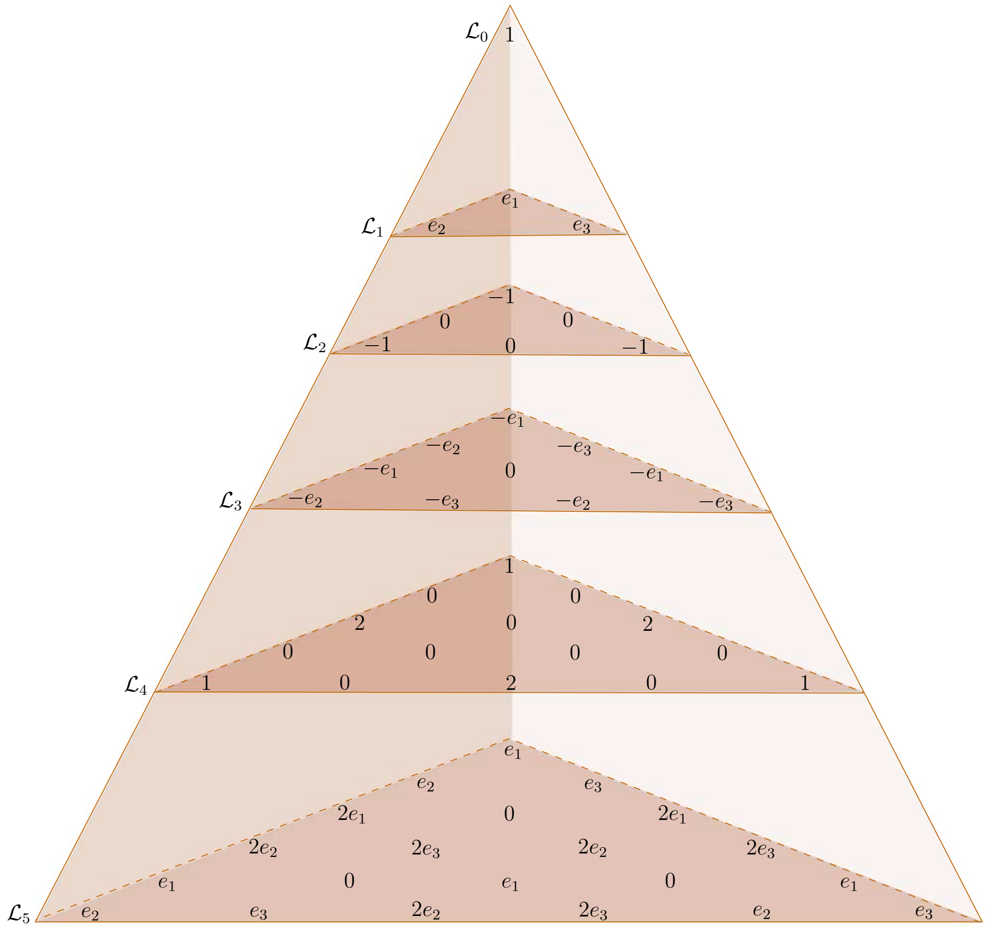

Figure 3.8: First layers of the hypercomplex Pascal tetrahedron

II. If k is odd then

$$\mathcal{E}_{(k,s_1,s_2)} = \begin{cases} (-1)^{\frac{k-1}{2}} \binom{\frac{k-1}{2}}{\frac{s_1}{2}} \binom{\frac{s_1}{2}}{\frac{s_2}{2}} e_1, & \text{if } s_1, s_2 \text{ even} \\ (-1)^{\frac{k-1}{2}} \binom{\frac{k-1}{2}}{\frac{s_1-1}{2}} \binom{\frac{s_1-1}{2}}{\frac{s_2}{2}} e_2, & \text{if } s_1 \text{ odd}, s_2 \text{ even} \\ (-1)^{\frac{k-1}{2}} \binom{\frac{k-1}{2}}{\frac{s_1-1}{2}} \binom{\frac{s_1-1}{2}}{\frac{s_2-1}{2}} e_3, & \text{if } s_1, s_2 \text{ odd} \\ 0, & \text{otherwise} \end{cases}. \tag{3.16}$$

Based on Figure 3.8, where the first layers of the hypercomplex Pascal tetrahedron are shown, it is possible to detect a different property of the such tetrahedron when comparing it with the Pascal tetrahedron with real entries. In this last case we have already noticed that the three facets of the Pascal tetrahedron coincide all with the Pascal 2-simplex, i.e. each facet is the Pascal triangle with real entries (see Figures 3.2 and 3.3). However, it is clear from Figure 3.8 that the three facets of the corresponding hypercomplex tetrahedron do not coincide. In fact, although one of the facets is the hypercomplex Pascal triangle, i.e. contains the coefficients in the expansion of $(e_1+e_2)^k$ for $k \geq 0$, the other two facets have the coefficients in the expansion of $(e_2+e_3)^k$ and $(e_3+e_1)^k$ (see Figure 3.9).

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & e_2 & & e_3 & & & \\
 & & -1 & & 0 & & -1 & & \\
 & -e_2 & & -e_3\;\;-e_2 & & & -e_3 & & \\
1 & & 0 & & 2 & & 0 & & 1 \\
e_2\;\;e_3 & & & 2e_2\;\;2e_3 & & & & e_2\;\;e_3 &
\end{array}
\qquad
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & e_3 & & e_1 & & & \\
 & & -1 & & 0 & & -1 & & \\
 & -e_3 & & -e_1\;\;-e_3 & & & -e_1 & & \\
1 & & 0 & & 2 & & 0 & & 1 \\
e_3\;\;e_1 & & & 2e_3\;\;2e_1 & & & & e_3\;\;e_1 &
\end{array}
$$

Figure 3.9: Facets of the Pascal tetrahedron with hypercomplex entries

It is also visible in Figure 3.8 the confirmation of Corollary 3.2.3. For example the sum of all elements of $\mathcal{L}_3$ corresponds to

$$-3(e_1+e_2+e_3) = (-3)^{\frac{3-1}{2}}(e_1+e_2+e_3) = \sum_{s_1=0}^{3}\sum_{s_2=0}^{s_1}\mathcal{E}_{(3,s_1,s_2)},$$

while for $\mathcal{L}_4$ we have

$$9 = (-3)^{\frac{4}{2}} = \sum_{s_1=0}^{4}\sum_{s_2=0}^{s_1}\mathcal{E}_{(4,s_1,s_2)}.$$

We recall also the identity of Corollary 3.2.4, in concrete for the case of $k=4$, i.e. $l=1$. It follows that

$$\sum_{s_1=0}^{4}\sum_{s_2=0}^{s_1}\binom{4}{s_1}\binom{s_1}{s_2}e_1^{4-s_1}\times e_2^{s_1-s_2}\times e_3^{s_2} = \sum_{s_1=0}^{2}\sum_{s_2=0}^{s_1}\binom{2}{s_1}\binom{s_1}{s_2} = 3^2 = 9. \tag{3.17}$$

In other words, (3.17) states that the sum of all numbers in the 4^{th} layer of the Pascal tetrahedron with hypercomplex entries coincides with the sum of all multinomial coefficients in the 2^{nd} layer of the Pascal tetrahedron with real entries (see Figures 3.2 and 3.8).

Finally we emphasize the $n=3$ case of Theorem 3.2.1. It allows to construct each layer of the Pascal tetrahedron with hypercomplex entries from the previous one.

$$\mathcal{E}_{(k,s_1,s_2)} = \mathcal{E}_{(k-1,s_1,s_2)}e_1 + \mathcal{E}_{(k-1,s_1-1,s_2)}e_2 + \mathcal{E}_{(k-1,s_1-1,s_2-1)}e_3, \tag{3.18}$$

where $\mathcal{E}_{(s_0,s_1,s_2)} = 0$ for $s_i > s_{i-1}$. The following figure exemplifies this recurrence relation between $\mathcal{L}_4$ and $\mathcal{L}_3$. The elements of $\mathcal{L}_3$ are inside rectangles, while those of $\mathcal{L}_3$ are marked in circles.

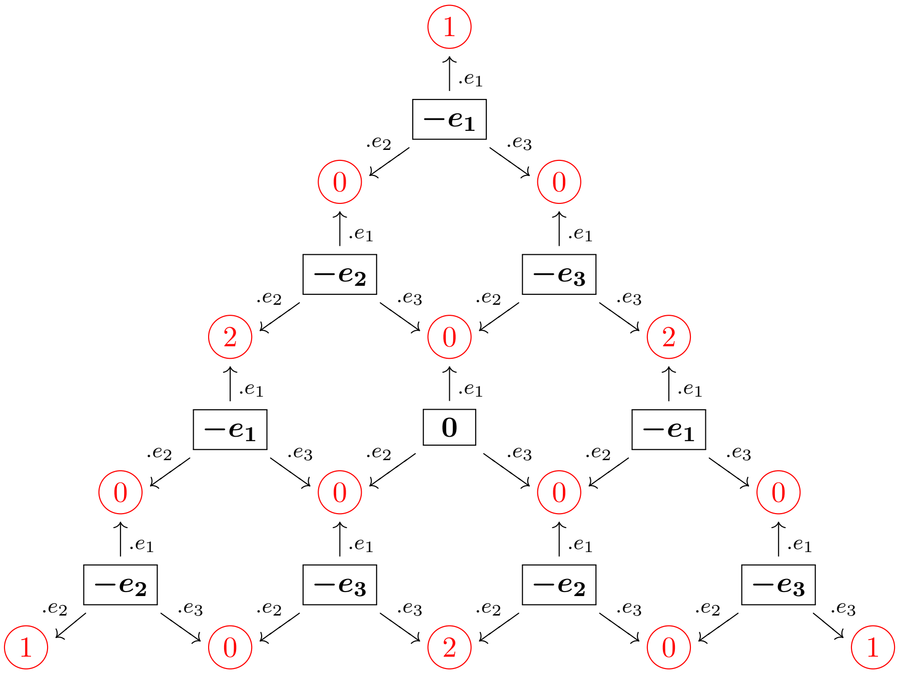

Figure 3.10: Relation between the elements of $\mathcal{L}_4$ and $\mathcal{L}_3$ in the hypercomplex Pascal tetrahedron

In what follows we illustrate the use of the recurrence relation (3.18) for the calculation of the second element of the fourth row of $\mathcal{L}_4$.

$$\begin{aligned}\mathcal{E}_{(4,3,1)} &= \binom{4}{3}\binom{3}{1} e_1 \times e_2^2 \times e_3 \\ &= \mathcal{E}_{(3,3,1)} e_1 + \mathcal{E}_{(3,2,1)} e_2 + \mathcal{E}_{(3,2,0)} e_3 = \boxed{-e_3}\, e_1 + \boxed{0}\, e_2 + \boxed{-e_1}\, e_3 = 0\end{aligned}$$

Comments to Chapter 3

A complete characterization of the hypercomplex Pascal tetrahedron was already published in [41]. The similarities with the real case motivated the consideration of arbitrary dimensions and the study of the essential but not trivial effects of the case of noncommutative multiplication. The result of this work can be seen as a contribution to generalize the well studied Pascal n-simplex with real entries and to derive new patterns in the hypercomplex context. At the same time it also constitutes a powerful tool for representing the coefficients of Appell polynomials with values in a Clifford Algebra.

Chapter 4

Pseudo-complex paravector-valued variables

4.1 Introduction

In Chapter 2 we used the concept of generalized Appell sequences, to study a class of holomorphic polynomials that behave like power functions under hypercomplex differentiation. In the complex case, the integer powers of the complex variable $z = x_0 + ix_1$ are holomorphic and at the same time Appell sequences with respect to the complex derivative $\partial = \frac{1}{2}(\partial_{x_0} - i\partial_{x_1})$. Here we consider polynomials in $(n+1)$ real variables that share those properties with z^k, but now with respect to the hypercomplex derivative $\partial = \frac{1}{2}(\partial_0 - \partial_{\underline{x}})$.

By analogy with the complex case, a natural candidate for extending z^k would be x^k, where $x = x_0 + \underline{x} \in \mathcal{A}_n$. However, since $\overline{\partial} x = (1-n)/2$ (cf. Proposition 1.2.8) the variable x is not Clifford holomorphic for $n > 1$ and therefore, one of the referred requirements is not fulfilled.

A second candidate could be the k-th power of the first degree polynomial of SAP (see Section 2.2), i.e., of

$$\mathcal{P}_1^n(x_0, \underline{x}) = x_0 + \frac{1}{n}\underline{x} = x_0 + \frac{1}{n}(x_1 e_1 + \cdots + x_n e_n).$$

But are all non-negative integer powers of $\mathcal{P}_1^n(x_0, \underline{x})$ also Clifford holomorphic or not? The answer is given by

Proposition 4.1.1 *If k is a non-negative integer, then the polynomial $\left(\mathcal{P}_1^n(x_0, \underline{x})\right)^k$ is holomorphic if and only if $n = 1$.*

Proof. For being short we split $g_k^{(n)}(x_0,\underline{x}) = \big(\mathcal{P}_1^n(x_0,\underline{x})\big)^k = \big(x_0 + \frac{1}{n}\underline{x}\big)^k$ into its scalar and vector part as

$$g_k^{(n)} = \sum_{s=0}^{\lfloor \frac{k}{2} \rfloor} \binom{k}{2s} x_0^{k-2s} \Big(\frac{1}{n}\underline{x}\Big)^{2s} + \sum_{s=0}^{\lfloor \frac{k-1}{2} \rfloor} \binom{k}{2s+1} x_0^{k-(2s+1)} \Big(\frac{1}{n}\underline{x}\Big)^{2s+1}. \tag{4.1}$$

Thus, by means of Lemma 1.2.7 follows easily

$$\begin{aligned}
\overline{\partial}\big(g_k^{(n)}\big) &= \partial_0 \operatorname{Sc}(g_k^{(n)}) + \partial_0 \operatorname{Vec}(g_k^{(n)}) + \partial_{\underline{x}} \operatorname{Sc}(g_k^{(n)}) + \partial_{\underline{x}} \operatorname{Vec}(g_k^{(n)}) \\
&= \sum_{s=0}^{\lfloor \frac{k}{2} \rfloor} \binom{k}{2s} \frac{k-2s}{n^{2s}} x_0^{k-2s-1} \underline{x}^{2s} + \sum_{s=0}^{\lfloor \frac{k-1}{2} \rfloor} \binom{k}{2s+1} \frac{k-2s-1}{n^{2s+1}} x_0^{k-2s-2} \underline{x}^{2s+1} \\
&- \sum_{s=1}^{\lfloor \frac{k}{2} \rfloor} \binom{k}{2s} \frac{2s}{n^{2s}} x_0^{k-2s} \underline{x}^{2s-1} - \sum_{s=0}^{\lfloor \frac{k-1}{2} \rfloor} \binom{k}{2s+1} \frac{n+2s}{n^{2s+1}} x_0^{k-2s-1} \underline{x}^{2s}.
\end{aligned}$$

Since $\binom{k}{0}k + \binom{k}{1}(-n)\frac{1}{n} = 0$, the coefficient of x_0^{k-1} vanishes and, consequently, the scalar part of $\overline{\partial} g_k^{(n)}$ is given by

$$\operatorname{Sc}\big(\overline{\partial} g_k^{(n)}\big) = \sum_{s=1}^{\lfloor \frac{k-1}{2} \rfloor} \left[\binom{k}{2s} \frac{k-2s}{n^{2s}} - \binom{k}{2s+1} \frac{n+2s}{n^{2s+1}} \right] x_0^{k-2s-1} \underline{x}^{2s}.$$

Simplifying the last expression we obtain

$$\operatorname{Sc}\big(\overline{\partial} g_k^{(n)}\big) = (n-1) \sum_{s=1}^{\lfloor \frac{k-1}{2} \rfloor} \frac{2s}{n^{2s+1}} \binom{k}{2s+1} x_0^{k-2s-1} \underline{x}^{2s}.$$

The same procedure shows that

$$\operatorname{Vec}\big(\overline{\partial} g_k^{(n)}\big) = (n-1) \sum_{s=1}^{\lfloor \frac{k}{2} \rfloor} \frac{2s}{n^{2s}} \binom{k}{2s} x_0^{k-2s} \underline{x}^{2s-1},$$

and therefore

$$\overline{\partial} g_k^{(n)} = \overline{\partial}\big(x_0 + \frac{1}{n}\underline{x}\big)^k = (n-1) \sum_{s=1}^{k-1} \frac{2}{n^{s+1}} \left\lfloor \frac{s+1}{2} \right\rfloor \binom{k}{s+1} x_0^{k-1-s} \underline{x}^s, \tag{4.2}$$

which proves the assertion, i.e. $\big(\mathcal{P}_1^n(x_0,\underline{x})\big)^k$ is holomorphic if and only if $n=1$. ∎

Proposition 4.1.1 shows again that the search for a linear hypercomplex function which could serve as a *Clifford holomorphic variable* with Clifford holomorphic powers is not so trivial as it seems to be. Indeed, in [47] R. Delanghe called attention to this problem and firstly introduced the concept of a *totally regular variable* (TRV) as being exactly of that desired type.

The aim of this chapter is to study sets of paravector valued homogeneous holomorphic polynomials which are powers of TRV and simultaneously possess the Appell property. In Section 4.2 we completely characterize, under some natural normalization condition, the set of all TRV defined in the three dimensional Euclidean space.

In Section 4.3 we construct analogously TRV in arbitrary dimensions which are isomorphic to the classical complex powers.

Section 4.4 considers powers of TRV that lead to new Appell sequences different from the SAP introduced in Chapter 2. Different parts of the results have already been published in [35] and [36].

4.2 Totally regular variables

In [47] R. Delanghe introduced for the first time the general concept of totally regular variables defined in $\mathbb{R}^n$ with values in $\mathcal{C}\ell_{0,n}$. In the context of $\mathcal{A}_n-$valued holomorphic functions the definition of Delanghe can be rewritten as:

Definition 4.2.1 *Consider $d_i \in \mathcal{A}_n$, $i = 0, \ldots, n$. A linear hypercomplex holomorphic function of the form*

$$z = z(x_0, \ldots, x_n) = x_0 d_0 + x_1 d_1 + \cdots + x_n d_n, \tag{4.3}$$

whose integer powers are also holomorphic is called totally regular variable *(TRV).*

In 1982, a special case of TRV were studied by Gürlebeck in [75]. He considered the case of quaternion valued variables in the form of

$$z = x_0 d_0 + x_1 d_1 + x_2 d_2 + x_3 d_3 \tag{4.4}$$

with $d_k = a_{k0} e_0 + a_{k1} e_1 + a_{k2} e_2 + a_{k3} e_1 e_2 \in \mathbb{H}$, $k = 0, 1, 2, 3$, not necessarily being linearly independent.

In order to characterize TRV of this form, Gürlebeck found necessary and sufficient conditions expressed by the rank of the matrix formed by the coefficients of the vector part of d_i

$$A = \begin{pmatrix} a_{01} & a_{02} & a_{03} \\ a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}. \tag{4.5}$$

The result is formulated in [75] as follows

Theorem 4.2.2 *If z is a quaternionic holomorphic variable of the form* (4.4), *then the following statements are equivalent*

1. *z is a TRV;*

2. *$d_k d_l = d_l d_k,\ k, l = 0, 1, 2, 3$;*

3. *The rank*[1] *of the matrix* (4.5) *is less than* 2. .

Remark 4.2.3 Due to its isomorphism with the complex variable $z = x_0 + ix_1$, each Fueter variable $z_s = x_s - x_0 e_s = (x_0 + e_s x_s)(-e_s),\ s = 1, 2, 3$ (cf.(1.8)) is a natural example of a TRV. In this case the conditions 2 and 3 of Theorem 4.2.2 are obviously fulfilled since for $s, l = 1, 2, 3,\ l \neq s$ we have $d_0 = -e_s; d_s = 1; d_l = 0$.

♦

We concentrate now on the general case of TRV defined in $\mathcal{A}_2$ and given in the same form

$$z = x_0 d_0 + x_1 d_1 + x_2 d_2 \in \mathcal{A}_2, \tag{4.6}$$

under the normalization condition

$$\partial z = d_0 = 1. \tag{4.7}$$

This normalization is motivated by the fact that the TRV we are looking for should also form an Appell sequence in the sense of Definition 2.1.1, as it occurs in $\mathbb{C}$ with $z = x_0 + ix_1$. Fixing the value of d_0 in (4.6) by 1 allows to prove an explicit representation theorem for those TRV defined in $\mathcal{A}_2$.

Theorem 4.2.4 *The set of all linear holomorphic variables $z = x_0 + x_1 d_1 + x_2 d_2$ which are TRV in the sense of Definition 4.2.1 consists of variables of the form*

$$z = x_0 + (i_1 x_1 + i_2 x_2)(i_1 e_1 + i_2 e_2), \tag{4.8}$$

with $(i_1, i_2) \in \mathbb{R}^2$ such that $i_1^2 + i_2^2 = 1$.

[1]In fact, excluding the trivial case that all d_i are zero, this means that the rank should be equal to 1.

Proof. Suppose $z = x_0 + x_1 d_1 + x_2 d_2$, where

$$d_1 = a_{10} + a_{11}e_1 + a_{12}e_2$$
$$d_2 = a_{20} + a_{21}e_1 + a_{22}e_2,$$

with $a_{mn} \in \mathbb{R}$, $m = 1, 2$ and $n = 0, 1, 2$. Since z is holomorphic it follows that

$$\overline{\partial} z = 1 + e_1 d_1 + e_2 d_2 = 0, \tag{4.9}$$

and therefore

$$\begin{cases} a_{10} = a_{20} = 0 \\ a_{12} = a_{21} \\ a_{11} + a_{22} = 1 \end{cases}. \tag{4.10}$$

Additionally, based on Theorem 4.2.2 (with $d_3 \equiv 0$), the reduced quaternion z is totally regular (and not only reduced to the real variable x_0) if and only if

$$\operatorname{rank} \begin{pmatrix} 0 & 0 & 0 \\ a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix} = \operatorname{rank} \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix} = 1, \tag{4.11}$$

or equivalently,

$$a_{11}a_{22} - a_{12}^2 = 0, \tag{4.12}$$

where a_{11} and a_{22} do not vanishing at the same time. Considering $\lambda := a_{12} = a_{21}$ as defined by a_{11} and a_{22}, the last relation is equivalent to

$$\lambda^2 = a_{11}a_{22}.$$

Two cases for the values of λ led now to the different possible cases of TRV $\in \mathcal{A}_2$ and their explicit representation.

Case A: $\lambda = 0$

If $a_{11} = 0$ then, from (4.10) and (4.12) we have that $a_{22} = 1$ and $a_{12} = a_{21} = 0$, respectively. Under these conditions the TRV obtained corresponds to the case $z = x_0 + x_2 e_2$. Analogously, if $a_{22} = 0$ we obtain also the corresponding case[2] of $z = x_0 + x_1 e_1$.

[2]In [58] these two cases of direct copies of the complex variable z (where the imaginary unit is replaced by one generator of $\mathcal{C}\ell_{0,n}$) are called "trivial" TRV.

Case B: $\lambda \neq 0$

This means that a_{11} and a_{22} have the same sign and together with (4.10) we conclude that a_{11} and a_{22} are both positive. Therefore we can write

$$a_{11} = i_1^2, \quad a_{22} = i_2^2 \quad \text{with} \quad i_1^2 + i_1^2 = 1. \tag{4.13}$$

As consequence follows $\lambda^2 = (i_1 i_2)^2$ and therefore also $a_{12}a_{21} = (i_1 i_2)^2$. Due to (4.10) we may now conclude that $a_{12} = a_{21} = \pm|i_1 i_2|$, depending from the choice of the square roots of a_{11} and a_{22} and their sign. Suppose that both have the same sign such that $a_{12} = a_{21} = |i_1 i_2| = i_1 i_2 > 0$. Then the general form of the associated TRV is given by

$$\begin{aligned} z &= x_0 + x_1(i_1^2 e_1 + i_1 i_2 e_2) + x_2(i_1 i_2 e_1 + i_2^2 e_2) \\ &= x_0 + (i_1 x_1 + i_2 x_2)(i_1 e_1 + i_2 e_2) = x_0 + (-i_1 x_1 - i_2 x_2)(-i_1 e_1 - i_2 e_2), \end{aligned} \tag{4.14}$$

while for the second case $a_{12} = a_{21} = -|i_1 i_2|$ we have

$$\begin{aligned} z &= x_0 + x_1(i_1^2 e_1 - |i_1 i_2| e_2) + x_2(-|i_1 i_2| e_1 + i_2^2 e_2) \\ &= x_0 + (|i_1| x_1 - |i_2| x_2)(|i_1| e_1 - |i_2| e_2) = x_0 + (-|i_1| x_1 + |i_2| x_2)(-|i_1| e_1 + |i_2| e_2). \end{aligned} \tag{4.15}$$

It is evident that both expressions (4.14) and (4.15) represent the same type of a TRV. The expression in (4.14) may be related to the unit vectors $(i_1, i_2) \in \mathbb{R}^2$ or $(-i_1, -i_2) \in \mathbb{R}^2$, while (4.15) may be related to the unit vectors $(|i_1|, -|i_2|) \in \mathbb{R}^2$ or $(-|i_1|, +|i_2|) \in \mathbb{R}^2$. ■

Remark 4.2.5 Suppose now that $(i_1, i_2) \in \mathbb{R}^2$ stands for an arbitrary unit vector. Since $(i_1 e_1 + i_2 e_2)^2 = -1$ due to $i_1^2 + i_2^2 = 1$ both TRV are obviously isomorphic to the complex variable $z = x + iy$. The fact that the TRV associated with the unit vector (i_1, i_2) is the same as the one associated with $(-i_1, -i_2)$ is equivalent to say that TRV associated with antipodal points on the unit circle S^1 are the same. Of course, the uniqueness of each variable is guaranteed if we restrict the set of unit vectors properly. In the following sections we will discuss this aspect in detail. ♦

Remark 4.2.6 If the normalization condition is changed to $d_0 = -e_1$, resp. $d_0 = -e_2$, then, instead of obtaining $e_1 z_1 = x_0 + x_1 e_1$ and $e_2 z_2 = x_0 + x_2 e_2$ as the special cases associated with $(\pm 1, 0)$ and $(0, \pm 1)$, resp., we would obtain the TRV $z_1 = x_1 - x_0 e_1$ and $z_2 = x_2 - x_0 e_2$, resp. ♦

To find an analogous explicit representation of TRV in the general case (4.3) for $n \geq 3$, i.e. for

$$z = x_0 d_0 + x_1 d_1 + \cdots + x_n d_n \in \mathcal{A}_n, \quad d_i = a_{i0} + a_{i1} e_1 + \cdots + a_{in} e_n, \tag{4.16}$$

it needs more advanced algebraic tools than the case $n = 2$. However, it is still possible to derive a (sufficient) condition in order to obtain a TRV.

Proposition 4.2.7 *Let z be a Clifford holomorphic variable of the form* (4.16) *where, like in [75], the rectangular matrix of the components of the vector parts of d_i, $i = 0, 1, \ldots, n$ is*

$$A = \begin{pmatrix} a_{01} & a_{02} & \ldots & a_{0n} \\ a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn}, \end{pmatrix}. \tag{4.17}$$

If $\operatorname{rank}(A) = 1$ *then there exists a pure vector $\underline{\delta} = \delta_1 e_1 + \cdots + \delta_n e_n$ such that z is a TRV of the form*

$$z = x_0 + (x_1\alpha_1 + \cdots + x_n\alpha_n)\underline{\delta}. \tag{4.18}$$

with $\alpha_i \in \mathbb{R}$, and $\alpha_i \neq 0$ for at least one $i = 1, \ldots, n$.

Proof. Indeed, being z Clifford holomorphic we have $\overline{\partial} z = d_0 + e_1 d_1 + \cdots + e_n d_n = 0$, and, together with the normalization condition $\partial z = 1$ (cf. (4.5)) it implies that $d_0 = 1$. This means in terms of the matrix (4.17) that the first row contains only zeros and that the square sub-matrix below the first row of A is symmetric and its trace is one:

$$\begin{cases} a_{01} = a_{02} = \cdots = a_{0n} = 0 \\ a_{ij} = a_{ji}, \quad i, j = 1, \ldots, n, i \neq j \\ a_{11} + a_{22} + \cdots + a_{nn} = 1. \end{cases}, \tag{4.19}$$

If now $\operatorname{rank}(A) = 1$ then there exists one vector $\underline{\delta} = \delta_1 e_1 + \cdots + \delta_n e_n \neq 0$ and $\alpha_i \neq 0$ for at least one $i = 1, \ldots, n$ that we can write

$$d_i = \alpha_i \underline{\delta}, \quad \alpha_i \in \mathbb{R}.$$

In such case we conclude that the variable (4.16) has the form

$$z = x_0 + \underline{\delta}(x_1\alpha_1 + \cdots + x_n\alpha_n) \tag{4.20}$$

and therefore $d_i d_j = d_j d_i$. To see that z is a TRV, it is enough to observe that, since any nonnegative integer power of z commutes with $\underline{\delta}$, it follows that

$$\begin{aligned} \overline{\partial} z^k &= k\big(z^{k-1} + e_1 z^{k-1}\alpha_1\underline{\delta} + \cdots + e_n z^{k-1}\alpha_n\underline{\delta}\big) \\ &= k\big(1 + e_1\alpha_1\underline{\delta} + \cdots + e_n\alpha_n\underline{\delta}\big) z^{k-1} \\ &= k\big(1 + e_1 d_1 + \cdots + e_n d_n\big) z^{k-1} = 0. \end{aligned}$$

■

Observe that formula (4.8) corresponds to the $n = 2$ case of (4.18) with $\alpha_1 = \delta_1 = i_1$ and $\alpha_2 = \delta_2 = i_2$, i.e., $\underline{\delta}^2 = -(i_1^2 + i_2^2) = -1$.

4.3 Generating pseudo-complex powers

Fueter's theorem (and its generalizations) is probably the most well-known approach for generating holomorphic functions by means of complex valued holomorphic functions f. However, the underlying use of the Laplace operator leads in general to Clifford holomorphic functions which don't preserve the structure of f.

The main goal of this section is to obtain TRV with values in $\mathcal{A}_n$ which share almost all properties of $z = x + iy \in \mathbb{C}$, including those related to its nonnegative integer powers. We follow here an idea of generating paravector-valued holomorphic functions defined in $\mathcal{A}_n$ by means of an appropriate substitution of variables in complex valued holomorphic functions which was proposed for the first time in [43].

Theorem 4.3.1 *Let $f(z) = u(x, y) + iv(x, y)$, $z = x + iy \in \mathbb{C}$ be a $\mathbb{C}$-holomorphic function defined in an open $G \subset \mathbb{C}$ and let $\vec{\imath} = (i_1, \ldots, i_n) \in S^{n-1}$. Then*

$$F(x) = u(x_0, x_{\hat{\imath}}) + \hat{\imath} v(x_0, x_{\hat{\imath}}), \tag{4.21}$$

where $\hat{\imath} = i_1 e_1 + \cdots + i_n e_n$ and $x_{\hat{\imath}} = i_1 x_1 + \cdots + i_n x_n$, is an $\mathcal{A}_n$-holomorphic function.

Proof. Since f is $\mathbb{C}$-holomorphic and $\frac{\partial x_{\hat{\imath}}}{\partial x_k} = i_k$ we have that

$$\begin{aligned}
(\partial_0 + \partial_{\underline{x}})F &= (\partial_0 + \partial_{\underline{x}}) \left(u(x_0, x_{\hat{\imath}}) + \hat{\imath} v(x_0, x_{\hat{\imath}}) \right) \\
&= \frac{\partial u(x_0, x_{\hat{\imath}})}{\partial x_0} + \left[\sum_{k=1}^{n} e_k \frac{\partial v(x_0, x_{\hat{\imath}})}{\partial x_k} \right] \hat{\imath} + \frac{\partial v(x_0, x_{\hat{\imath}})}{\partial x_0} \hat{\imath} + \left[\sum_{k=1}^{n} e_k \frac{\partial u(x_0, x_{\hat{\imath}})}{\partial x_k} \right] \\
&= \frac{\partial u(x_0, x_{\hat{\imath}})}{\partial x_0} + \frac{\partial v(x_0, x_{\hat{\imath}})}{\partial x_{\hat{\imath}}} \left[\sum_{k=1}^{n} e_k \frac{\partial x_{\hat{\imath}}}{\partial x_k} \right] \hat{\imath} + \frac{\partial v(x_0, x_{\hat{\imath}})}{\partial x_0} \hat{\imath} + \frac{\partial u(x_0, x_{\hat{\imath}})}{\partial x_{\hat{\imath}}} \left[\sum_{k=1}^{n} e_k \frac{\partial x_{\hat{\imath}}}{\partial x_k} \right] \\
&= 0,
\end{aligned}$$

and the result follows. ∎

This method differs from Fueter's approach in the sense that this one consists on the substitution of the imaginary complex unit i by a *constant* pure unit vector $\hat{\imath}$, while Fueter's approach is realized by the substitution of i by the *variable* unit vector $\omega(\underline{x}) = \frac{\underline{x}}{|\underline{x}|}$. Moreover, in the first method x and y are replaced by x_0 and $x_{\hat{\imath}} = i_1 x_1 + \cdots + i_n x_n$, respectively, while in the second method x and y are replaced by x_0 and $|\underline{x}|$. Finally, we underline the fact that the obtained function F is itself a paravector-valued holomorphic function that preserves the structure of f. Theorem 1.2.11 shows that Fueter's approach involves additionally a diferential operator of higher order to obtain a Clifford holomorphic function.

The application of Theorem 4.3.1 to the identity function $f(z) = z = x + iy$ gives the following new type of variables which will play an important role in this work

Definition 4.3.2 *Consider* $\vec{\imath} = (i_1, \ldots, i_n) \in S^{n-1}$, *its scalar product with* $\vec{x} \in \mathbb{R}^n$ *in the form* $x_{\hat{\imath}} = i_1 x_1 + \cdots + i_n x_n$, *and the associated hypercomplex pure vector* $\hat{\imath} = i_1 e_1 + \cdots + i_n e_n$. *The linear function* $Z_{\vec{\imath}} : \mathcal{A}_n \longmapsto \mathbb{C}_{\hat{\imath}}$, *with* $\mathbb{C}_{\hat{\imath}} = \mathrm{span}_{\mathbb{R}}\{1, \hat{\imath}\} \subset \mathcal{C}\ell_{0,n}$ *and given by*

$$Z_{\vec{\imath}} = Z_{\vec{\imath}}\,(x_0, \vec{x}) = x_0 + \hat{\imath} x_{\hat{\imath}} = x_0 + (i_1 e_1 + \cdots + i_n e_n)(i_1 x_1 + \cdots + i_n x_n) \tag{4.22}$$

defines a variable which we designate by ***pseudo-complex variable*** *(PCV).*

Note that the designation of these variables should include the dimension n of the space and, consequently, a more complete notation could be $Z_{\vec{\imath}^{(n)}}$. If the dimension is clearly specified, we use for simplicity $Z_{\vec{\imath}}$ instead of $Z_{\vec{\imath}^{(n)}}$.

It follows immediately from the definition that for a vector $\vec{u}_k$ which has 1 in the k-th coordinate and 0's elsewhere, i.e. $\vec{u}_k = (0, \ldots, 1, \ldots, 0)$, the corresponding PCV is

$$Z_{\vec{u}_k} = x_0 + x_k e_k = e_k z_k.$$

Observe also that the form of the PCV given by (4.22) coincides with the special case of (4.18) when considering $\delta_s = \alpha_s = i_s$ and $\underline{\delta}^2 = -|\underline{\delta}|^2 = -1$, $s = 1, \ldots, n$. Moreover, when $n = 2$, the PCV (4.22) reduces to the TRV (4.8) introduced in the previous section.

The used designation has its origin in the identification of

$$z = x + iy \in \mathbb{C} \qquad \text{and} \qquad Z_{\vec{\imath}} = x_0 + \hat{\imath} x_{\hat{\imath}} \in \mathcal{A}_n, \tag{4.23}$$

i.e. in the isomorphism between z and $Z_{\vec{\imath}}$ given by $\varphi : \mathbb{C} \longmapsto \mathcal{A}_n$ with $\varphi(x) = x_0, \varphi(i) = \hat{\imath}$ and $\varphi(y) = x_{\hat{\imath}}$. Moreover, since the hypercomplex vector $\hat{\imath}$ verifies $\hat{\imath}^2 = -1$ and $x_{\hat{\imath}}$ is real valued we can benefit from the properties of z. Of course, in the complex case the only possible choices for $\vec{\imath} = i_1$ are the values -1 and $+1$. Hence, we obtain for these two cases the same variable which is given by

$$z = x_0 + (1e_1)(1x_1) = x_0 + (-1e_1)(-1x_1) = x_0 + x_1 e_1. \tag{4.24}$$

In $\mathcal{A}_n$ the unit vector $\vec{\imath}$ has n components which can be identified with the components of a vector on the unit sphere S^{n-1}. Analogously to (4.24), we should be aware of the fact that $Z_{\vec{\imath}_s} = Z_{\vec{\imath}_t}$ is possible, with $\vec{\imath}_s \neq \vec{\imath}_t$. As an example we refer to the case of $n = 2$ with $\vec{\imath}_s = \big(\cos\frac{\pi}{4}, \sin\frac{\pi}{4}\big)$ and $\vec{\imath}_t = \big(-\cos\frac{\pi}{4}, -\sin\frac{\pi}{4}\big)$. In this case one has

$$Z_{\vec{\imath}_s} = Z_{\vec{\imath}_t} = x_0 + \frac{1}{2}(x_1 + x_2)(e_1 + e_2).$$

In general, for any two antipodal points on S^{n-1} the PCV associated with the corresponding unit vectors coincide. Based on this, we restrict the set of unit vectors to a subset of vectors on a unitary semi-sphere, to ensure that for $\vec{\imath}_s \neq \vec{\imath}_t$ also $Z_{\vec{\imath}_s} \neq Z_{\vec{\imath}_t}$.

In the following, two examples of different choices for the unit vector $\vec{\imath} = (i_1, \ldots, i_n)$ are shown.

1. In terms of spherical coordinates:

$$\begin{aligned}
i_1 &= \cos\phi_1 \\
i_2 &= \sin\phi_1 \cos\phi_2 \\
i_3 &= \sin\phi_1 \sin\phi_2 \cos\phi_3 \\
&\vdots \\
i_{n-1} &= \sin\phi_1 \sin\phi_2 \ldots \sin\phi_{n-2} \cos\phi_{n-1} \\
i_n &= \sin\phi_1 \sin\phi_2 \ldots \sin\phi_{n-2} \sin\phi_{n-1},
\end{aligned} \tag{4.25}$$

with $\phi_1, \ldots, \phi_{n-1}$ such that $\phi_{n-1} \in [0, 2\pi[$ and $\phi_i \in [0, \pi]$, $i = 1, \ldots, n-2$.

2. In terms of:

$$\vec{\imath} = (i_1, \ldots, i_n) = \left(\frac{1}{\sqrt{1 + |a|^2}}, \frac{a_1}{\sqrt{1 + |a|^2}}, \ldots, \frac{a_{n-1}}{\sqrt{1 + |a|^2}} \right), \tag{4.26}$$

where $\vec{a} = (a_1, \ldots, a_{n-1}) \in \mathbb{R}^{n-1}$ with $|a|^2 = a_1^2 + \cdots + a_{n-1}^2$.

There are many other choices for $\vec{\imath}$ and, in general, the use of a certain form will depend on the problem we are dealing with. In addition, each PCV (4.22) admits also different representations which follow immediately from its *complex structure*:

$$\begin{aligned}
Z_{\vec{\imath}} &= x_0 + \hat{\imath} x_{\hat{\imath}} &(4.27)\\
&= -\hat{\imath}^2 x_0 + \hat{\imath} x_{\hat{\imath}} = \hat{\imath}(x_{\hat{\imath}} - x_0 \hat{\imath}) \\
&= \hat{\imath}(z_1 i_1 + \cdots + z_n i_n). &(4.28)
\end{aligned}$$

Since $\vec{\imath} = (i_1, \ldots, i_n) \in \mathbb{R}^n \subset \mathcal{H}^n$, we can rewrite this last expression in terms of an inner product defined in $\mathcal{H}^n$. In concrete, for $\vec{z} = (z_1, \ldots, z_n) \in \mathcal{H}^n$ we have

$$Z_{\vec{\imath}} = \hat{\imath} \big\langle (z_1, \ldots, z_n), (i_1, \ldots, i_n) \big\rangle.$$

For $n = 1$, last expression reduces again to

$$Z_{\vec{\imath}} = i_1 e_1 \langle z_1, i_1 \rangle = i_1^2 e_1 z_1 = x_0 + x_1 e_1. \tag{4.29}$$

We call attention to both representations (4.27) and (4.28) of the PCV $Z_{\vec{\imath}}$. While in the first case its character of paravector is reflected, the second stresses its connection with the Fueter variables z_k.

Corollary 4.3.3 *Consider $\hat{\imath} = i_1 e_1 + \cdots + i_n e_n$, with $\hat{\imath}^2 = -1$ and $x_{\hat{\imath}} = i_1 x_1 + \cdots + i_n x_n$, then $Z_{\vec{\imath}} = x_0 + \hat{\imath} x_{\hat{\imath}}$ is a totally regular variable.*

Proof. Applying Theorem 4.3.1 to the standard complex powers $z^k = (x+iy)^k$, $k \in \mathbb{N}$ leads to

$$(x_0 + \hat{\imath} x_{\hat{\imath}})^k = (Z_{\vec{\imath}})^k, \tag{4.30}$$

and therefore $Z_{\vec{\imath}}$ is a TRV. ■

Based on this we propose the following

Definition 4.3.4 *Consider $\hat{\imath}$ and $x_{\hat{\imath}}$ in the conditions of Definition 4.3.2. The **pseudo-complex power of order k associated with** $\hat{\imath}$ is defined as*

$$Z_{\vec{\imath}}^k = (Z_{\vec{\imath}})^k = (x_0 + \hat{\imath} x_{\hat{\imath}})^k. \tag{4.31}$$

Since different problems frequently demand different representations, in the following we present two new forms of pseudo-complex powers (PCP).

1. In terms of Fueter variables:

$$Z_{\vec{\imath}}^k = \hat{\imath}^k (z_1 i_1 + \cdots + z_n i_n)^k. \tag{4.32}$$

2. In terms of scalar and vector part:

$$\begin{aligned} Z_{\vec{\imath}}^k &= \sum_{l=0}^{k} \binom{k}{l} x_0^{k-l} (x_{\hat{\imath}} \hat{\imath})^l \\ &= \sum_{l=0}^{\left[\frac{k}{2}\right]} (-1)^l \binom{k}{2l} x_0^{k-2l} (x_{\hat{\imath}})^{2l} + \hat{\imath} \sum_{l=0}^{\left[\frac{k-1}{2}\right]} (-1)^l \binom{k}{2l+1} x_0^{k-2l-1} (x_{\hat{\imath}})^{2l+1} \end{aligned} \tag{4.33}$$

4.4 Pseudo-complex powers as Appell polynomials

Pseudo-complex powers have already been considered in [30], where the authors proved the following essential result.

Theorem 4.4.1 (Theorem 1, [30]) *For each* $\hat{\imath} = i_1 e_1 + \cdots + i_n e_n$*, such that* $\hat{\imath}^2 = -1$*, the sequence* $\left(Z_{\hat{\imath}}^k\right)_{k \geq 0}$ *is an Appell set.*

Besides emphasizing the isomorphism between the ordinary complex powers z^k and the PCP $Z_{\hat{\imath}}^k$, this result allows to conclude that both the SAP and the PCP are examples of sequences of Appell polynomials in $(n+1)$ variables. Moreover, both sequences share the property that their restriction to the real line gives the ordinary real Appell sequence $\left(x_0^k\right)_{k \geq 0}$. This observation motivated the following question:

Are the SAP and the PCP the only type of paravector valued Appell polynomials such that their restriction to $\underline{x} = 0$ *gives* x_0^k*?*

In order to study this problem we use the fact that generalized Appell sequences of holomorphic polynomials in the setting of hypercomplex function theory also satisfy a corresponding binomial type theorem (cf. Theorem 2.2.7). Therefore, we consider homogeneous polynomials given by

$$P_k(x_0, \underline{x}) = \sum_{s=0}^{k} \binom{k}{s} d_s(n) x_0^{k-s} \underline{X}^s, \tag{4.34}$$

where $d_s(n)$ are suitably chosen real coefficients and

$$\underline{X} = X_1(x_1, \ldots, x_n) e_1 + \cdots + X_n(x_1, \ldots, x_n) e_n, \tag{4.35}$$

is a pure vector valued function such that $X_j = X_j(x_1, \ldots, x_n)$ are real valued linear functions for $j = 1, \ldots, n$. It is easy to recognize that the polynomials (4.34) have values in $\mathcal{A}_n$ since $\underline{X}^s$ is real valued or pure vector valued, depending whether s is even or odd, respectively. Thus, as already mentioned, left holomorphic functions of the form (4.34) are also right holomorphic and, consequently, it is enough to solve the real system of first order partial differential equations corresponding to $\overline{\partial} P_k = 0$. Moreover, with the additional assumption $d_0(n) \equiv 1$ we can guarantee the normalization condition $P_k(x_0, 0) = x_0^k$.

Recently, it has been obtained a complete characterization of polynomials of the form (4.34) for the case of three real variables. Note that in this case P_1 is holomorphic if and only if it is of the form

$$P_1(x_0, x_1 e_1 + x_2 e_2) = x_0 + d_1\big((a_1 x_1 + \lambda_1 x_2) e_1 + (\lambda_1 x_1 + a_2 x_2) e_2\big), \tag{4.36}$$

for real parameters a_1, a_2 and λ_1 such that

$$d_1 = \frac{1}{a_1 + a_2}, \qquad \text{for} \qquad a_1 + a_2 \neq 0. \tag{4.37}$$

Having this in consideration, the authors of [58] proved the following result:

Theorem 4.4.2 (Theorem 2.1, [58]) *For polynomials of the form*

$$P_k(x_0, \underline{x}) = \sum_{s=0}^{k} \binom{k}{s} d_s x_0^{k-s} (X_1 e_1 + X_2 e_2)^s$$

there are exactly two different types of non-trivial Appell polynomials, namely

1. *The SAP* (2.11)

$$P_k(x_0, \underline{x}) = \mathcal{P}_k^n(x_0, \underline{x}) = \sum_{s=0}^{k} \binom{k}{s} c_s(2) x_0^{k-s} \underline{x}^s.$$

2. *The PCP* (4.31)

$$P_k(x_0, \underline{x}) = Z_{\vec{\imath}}^k = \sum_{s=0}^{k} \binom{k}{s} x_0^{k-s} (i_1 x_1 + i_2 x_2)^s \hat{\imath}^s,$$

where $\hat{\imath}^2 = \big(i_1 e_1 + i_2 e_2\big)^2 = -1.$

Remark 4.4.3 According to (4.37), if $a_1 a_2 = 0$, then only one of the parameters a_i vanishes. Assuming that $a_1 = 0$, and therefore $d_1 = \frac{1}{a_2}$, one obtains in (4.36) the polynomial

$$P_1(x_0, \underline{x}) = x_0 + \Big(\tfrac{\lambda_1}{a_2} x_2 e_1 + \big(\tfrac{\lambda_1}{a_2} x_1 + x_2\big) e_2\Big).$$

If we proceed by considering P_2 then we conclude easily that this polynomial is holomorphic if and only if $\lambda_1 = 0$, which means that, in such case we end up with the trivial Appell polynomials

$$P_k(x_0, \underline{x}) = P_k(x_0, x_2 e_2) = \sum_{s=0}^{k} \binom{k}{s} x_0^{k-s} (x_2 e_2)^s = (x_0 + x_2 e_2)^k.$$

Analogously, if $a_2 = 0$ it follows that

$$P_k(x_0, \underline{x}) = P_k(x_0, x_1 e_1) = \sum_{s=0}^{k} \binom{k}{s} x_0^{k-s} (x_1 e_1)^s = (x_0 + x_1 e_1)^k,$$

that is, both cases can be identified with two copies of the ordinary complex power function $z^k = (x + iy)^k$ after setting $x_0 = x$ and $e_j = i, x_j = y, \; j = 1, 2$. These cases are trivial in the sense that $e_1 \partial_1 + e_2 \partial_2$ acts only as differential operator with respect to x_1 or x_2 if the components of X are not depending on x_2 or x_1, respectively. ♦

Theorem 4.4.2 shows the exclusive role played by the SAP and the PCP in the context of Appell polynomials in $\mathbb{R}^3$ restricted to the condition $d_0(2) = 1$. We focus now on the extension of this result to the $n = 3$ form of the polynomials (4.34), i.e.

$$P_k(x_0, \underline{x}) = \sum_{s=0}^{k} \binom{k}{s} d_s(3) x_0^{k-s} (X_1 e_1 + X_2 e_2 + X_3 e_3)^s. \tag{4.38}$$

From now on we use briefly $d_s = d_s(3), s \geq 0$. In this case the first degree polynomial $P_1(x_0, \underline{x}) = x_0 + d_1 \underline{X}$ is holomorphic if the real linear functions X_j are of the form

$$X_1 = a_1 x_1 + \lambda_1 x_2 + \lambda_2 x_3, \quad X_2 = \lambda_1 x_1 + a_2 x_2 + \lambda_3 x_3, \quad X_3 = \lambda_2 x_1 + \lambda_3 x_2 + a_3 x_3, \tag{4.39}$$

for real parameters $a_1, a_2, a_3, \lambda_1, \lambda_2$ and λ_3 such that the coefficient d_1 is given by

$$d_1 = \frac{1}{a_1 + a_2 + a_3}, \qquad \text{for} \quad a_1 + a_2 + a_3 \neq 0. \tag{4.40}$$

It follows immediately that the special cases

$$\lambda_1 = \lambda_2 = a_1 = 0 \qquad \text{or} \qquad \lambda_1 = \lambda_3 = a_2 = 0 \qquad \text{or} \qquad \lambda_2 = \lambda_3 = a_3 = 0 \tag{4.41}$$

correspond to 3D polynomials and therefore Theorem 4.4.2 holds, i.e. (4.38) are, in fact, 3D standard Appell polynomials, 3D pseudo-complex polynomials or complex powers.

We proceed now by examining under what conditions the second degree polynomial $P_2(x_0, \underline{x}) = x_0 + 2d_1 \underline{X} + d_2 \underline{X}^2$ is holomorphic. This problem is equivalent, for independent x_1, x_2 and x_3, to solve the nonlinear system

$$\begin{cases} a_1 - d_2(a_1^2 + \lambda_1^2 + \lambda_2^2)(a_1 + a_2 + a_3) = 0 \\ a_2 - d_2(\lambda_1^2 + a_2^2 + \lambda_3^2)(a_1 + a_2 + a_3) = 0 \\ a_3 - d_2(\lambda_2^2 + \lambda_3^2 + a_3^2)(a_1 + a_2 + a_3) = 0 \\ \lambda_1 - d_2(\lambda_1(a_1 + a_2) + \lambda_2\lambda_3)(a_1 + a_2 + a_3) = 0 \\ \lambda_2 - d_2(\lambda_2(a_1 + a_3) + \lambda_1\lambda_3)(a_1 + a_2 + a_3) = 0 \\ \lambda_3 - d_2(\lambda_3(a_2 + a_3) + \lambda_1\lambda_2)(a_1 + a_2 + a_3) = 0 \end{cases} \tag{4.42}$$

Cumbersome, but straightforward calculations lead to the following cases:

1. $\lambda_1 = \lambda_2 = \lambda_3 = 0$

We can assume that $a_1 a_2 a_3 \neq 0$, since the trivial case where one of the a_i's is zero was already considered in (4.41). The solution of (4.42) is:

$$a_1 = a_2 = a_3 = \alpha \ \ (\alpha \neq 0) \qquad \text{and} \qquad d_2 = \frac{1}{3\alpha^2}. \tag{4.43}$$

In this case, since $d_1 = \frac{1}{3\alpha}$ it follows immediately that $P_1(x_0, \underline{x}) = \mathcal{P}_1(x) = x_0 + \frac{1}{3}\underline{x}$. Furthermore, $\underline{X} = \alpha(x_1e_1 + x_2e_2 + x_3e_3) = \alpha\underline{x}$, which allows to express P_k as

$$P_k(x_0, \underline{x}) = \sum_{s=0}^{k} \binom{k}{s} d_s \alpha^s x_0^{k-s} \underline{x}^s.$$

If we now apply Theorem 2 of [28], where homogeneous holomorphic polynomials expressed in terms of powers of x_0 and $\underline{x}$ where completely characterized, then we conclude that

$$d_k = \frac{c_k(3)}{\alpha^k}.$$

In other words, this case corresponds to the SAP (cf (2.11)) in $\mathbb{R}^4$.

2. $\lambda_1 \neq 0$ or $\lambda_2 \neq 0$ or $\lambda_3 \neq 0$

These cases correspond to assuming that either one of the parameters $\lambda_k, k = 1, 2, 3$ is nonzero and the other two vanish or all the parameters are nonzero. The situation where two of the parameters are nonzero and the other one is zero is not possible (see equations 4-6 of (4.42)).

If one of the parameters is nonzero and the other two vanish, the system (4.42) simplifies and can be solved easily with respect to d_2. It leads to 3D polynomials as one of the cases of (4.41) (where the corresponding coefficient $a_k, k = 1, 2, 3$ is assumed of being equal to zero) or to a solution of the form $d_2 = 2d_1^2$ (where the corresponding coefficient $a_k, k = 1, 2, 3$ is not zero). Because of $d_2 = 2d_1^2$ the binomial expansion of the Appell polynomial corresponds to a particular case of a 4D polynomial that will be treated next in **ii.**

In the case $\lambda_1\lambda_2\lambda_3 \neq 0$, the system (4.42) leads to the following quadratic equation in d_2.

$$(a_1 + a_2 + a_3)^4 d_2^2 - 3(a_1 + a_2 + a_3)^2 d_2 + 2 = 0,$$

which admits two different solutions:

$$\textbf{i. } d_2 = d_1^2 \qquad \text{or} \qquad \textbf{ii. } d_2 = 2d_1^2.$$

In what follows we analyze both cases separately.

i. In this case $d_2 = \frac{1}{(a_1+a_2+a_3)^2} = d_1^2$ and $\lambda_1,\ \lambda_2,\ \lambda_3$ are roots of

$$\lambda_1^2 = a_1a_2, \qquad \lambda_2^2 = a_1a_3, \qquad \lambda_3^2 = a_2a_3, \tag{4.44}$$

chosen so that $\lambda_1\lambda_2\lambda_3 = a_1a_2a_3$. We point out that, due to (4.44), if a_1, a_2 and a_3 are nonzero, then they must have the same sign. Following the technique of [58], we introduce now the real parameters

$$i_1^2 = \tfrac{a_1}{a_1+a_2+a_3}, \qquad i_2^2 = \tfrac{a_2}{a_1+a_2+a_3}, \qquad i_3^2 = \tfrac{a_3}{a_1+a_2+a_3}, \tag{4.45}$$

which allow to write

$$P_1(x_0, \underline{x}) = x_0 + (i_1x_1 + i_2x_2 + i_3x_3)\,(i_1e_1 + i_2e_2 + i_3e_3)\,,$$

for a chosen triplet of those roots i_1, i_2 and i_3 as defined in (4.45). Notice that from this relation we obtain $i_1^2+i_2^2+i_3^2 = 1$ and it follows immediately that $(i_1e_1+i_2e_2+i_3e_3)^2 = -1$. In other words, we obtain the 4D-version of the PCP (4.31), if $\lambda_1\lambda_2\lambda_3 \neq 0$ and the 3D-version, in the other cases.

ii. In this situation we have that $d_2 = \frac{2}{(a_1+a_2+a_3)^2} = 2d_1^2$ and $\lambda_1,\ \lambda_2,\ \lambda_3$ are roots of

$$\lambda_1^2 = \frac{1}{4}A_1A_2, \qquad \lambda_2^2 = \frac{1}{4}A_1A_3, \qquad \lambda_3^2 = \frac{1}{4}A_2A_3, \tag{4.46}$$

where $A_1 = a_2 + a_3 - a_1$, $A_2 = a_1 + a_3 - a_2$ and $A_3 = a_1 + a_2 - a_3$ are real parameters which must have the same sign, if none of them vanishes. In this case, the roots λ_i must be chosen in order to verify the relation

$$-8\lambda_1\lambda_2\lambda_3 = A_1A_2A_3. \tag{4.47}$$

We proceed by adapting the procedure used in case **2.i**. According to (4.46), if A_1, A_2 and A_3 are non-zero then they must have the same sign, which allows us to define the real parameters

$$j_1^2 = \tfrac{A_1}{A_1+A_2+A_3}, \qquad j_2^2 = \tfrac{A_2}{A_1+A_2+A_3}, \qquad j_3^2 = \tfrac{A_3}{A_1+A_2+A_3} \tag{4.48}$$

which verify the relation $j_1^2 + j_2^2 + j_3^2 = 1$. By observing that $A_1 + A_2 + A_3 = a_1 + a_2 + a_3$ we conclude from (4.46) and (4.48) that

$$\lambda_1 = \pm\frac{1}{2}|a_1 + a_2 + a_3||j_1j_2|, \quad \lambda_2 = \pm\frac{1}{2}|a_1 + a_2 + a_3||j_1j_3|, \quad \lambda_3 = \pm\frac{1}{2}|a_1 + a_2 + a_3||j_2j_3|.$$

In addition, we note that, according to relation (4.47), $\lambda_1\lambda_2\lambda_3$ has the opposite sign of $A_1A_2A_3$ and consequently the opposite sign of $a_1 + a_2 + a_3$. Thus, independently of the sign of j_1, j_2 and j_3 the following situations may be considered. If $a_1 + a_2 + a_3 > 0$ then $\lambda_1\lambda_2\lambda_3 < 0$, that is, $\lambda_l < 0$ for all $l \in \{1, 2, 3\}$ or $\lambda_l < 0$ for only one $l \in \{1, 2, 3\}$ and $\lambda_k > 0$ for $k \neq l$. Analogously, if $a_1 + a_2 + a_3 < 0$ then $\lambda_l > 0$ for all $l \in \{1, 2, 3\}$ or $\lambda_l > 0$ for only one $l \in \{1,2,3\}$ and $\lambda_k < 0$ for $k \neq l$. For simplicity, from now on we suppose that $j_1, j_2, j_3, a_1+a_2+a_3 > 0$ and that $\lambda_1, \lambda_2, \lambda_3 < 0$, i.e. we will assume

$$\lambda_1 = -\frac{1}{2}(a_1+a_2+a_3)j_1j_2, \quad \lambda_2 = -\frac{1}{2}(a_1+a_2+a_3)j_1j_3, \quad \lambda_3 = -\frac{1}{2}(a_1+a_2+a_3)j_2j_3.$$

By observing that

$$\frac{a_l}{a_1+a_2+a_3} = \frac{1}{2}(1-j_l^2), \quad l \in \{1,2,3\},$$

it is possible to express P_1 as

$$\begin{aligned} P_1(x_0, \underline{x}) = x_0 + \frac{1}{2}\Big[&\big((1-j_1^2)x_1 - j_1j_2x_2 - j_1j_3x_3\big)e_1 \\ &+ \big(-j_1j_2x_1 + (1-j_2^2)x_2 - j_2j_3x_3\big)e_2 \\ &+ \big(-j_1j_3x_1 - j_2j_3x_2 + (1-j_3^2)x_3\big)e_3\Big]. \end{aligned} \tag{4.49}$$

Equivalently, one can write

$$P_1(x_0, \underline{x}) = x_0 + \tfrac{1}{2}(\underline{x} - \underline{\tilde{X}}), \tag{4.50}$$

where $\underline{\tilde{X}}$ denotes the vector part of a pseudo-complex first degree polynomial, i.e.

$$\underline{\tilde{X}} = (j_1x_1 + j_2x_2 + j_3x_3)\,(j_1e_1 + j_2e_2 + j_3e_3)$$

and $\underline{X} = \frac{1}{2d_1}(\underline{x} - \underline{\tilde{X}})$.

Our aim now is to obtain the general form of the coefficients d_s given in (4.38). Before we continue we prove an auxiliary result which will play an import role in the subsequent work.

Lemma 4.4.4 *For a nonnegative integer k the following relation holds*

$$\partial_{\underline{x}}\big(\underline{x} - \underline{\tilde{X}}\big)^k = \begin{cases} -k\big(\underline{x} - \underline{\tilde{X}}\big)^{k-1}, & k \text{ even} \\ -(k+1)\big(\underline{x} - \underline{\tilde{X}}\big)^{k-1}, & k \text{ odd} \end{cases} \tag{4.51}$$

Proof. By recalling (4.49) and (4.50) we have that $\underline{x} - \underline{\tilde{X}} = w_1e_1 + w_2e_2 + w_3e_3$, where $w_i = w_i(x_1, x_2, x_3)$ are real valued linear functions given by

$$\begin{aligned} w_1 &= (1-j_1^2)x_1 - j_1j_2x_2 - j_1j_3x_3 \\ w_2 &= -j_1j_2x_1 + (1-j_2^2)x_2 - j_2j_3x_3 \\ w_3 &= -j_1j_3x_1 - j_2j_3x_2 + (1-j_3^2)x_3. \end{aligned}$$

If k is even then both $(\underline{x}-\underline{\tilde{X}})^k$ and $(\underline{x}-\underline{\tilde{X}})^{k-2}$ are real valued homogeneous polynomials and therefore

$$\begin{aligned}\partial_{\underline{x}}(\underline{x}-\underline{\tilde{X}})^k &= \frac{k}{2}(\underline{x}-\underline{\tilde{X}})^{k-2}\partial_{\underline{x}}\big(-w_1^2-w_2^2-w_3^2\big)\\ &= k(\underline{x}-\underline{\tilde{X}})^{k-2}\big(-(w_1e_1+w_2e_2+w_3e_3)+(w_1j_1+w_2j_2+w_3j_3)\big).\end{aligned}$$

Thus, after verifying that $w_1j_1+w_2j_2+w_3j_3=0$ the assertion follows immediately.

When k is odd one has that $(\underline{x}-\underline{\tilde{X}})^{k-1}$ is real valued which leads to

$$\begin{aligned}\partial_{\underline{x}}(\underline{x}-\underline{\tilde{X}})^k &= \Big[\partial_{\underline{x}}(\underline{x}-\underline{\tilde{X}})^{k-1}\Big](\underline{x}-\underline{\tilde{X}})+(\underline{x}-\underline{\tilde{X}})^{k-1}\Big[\partial_{\underline{x}}(\underline{x}-\underline{\tilde{X}})\Big]\\ &= -(k-1)(\underline{x}-\underline{\tilde{X}})^{k-2}(\underline{x}-\underline{\tilde{X}})-2(\underline{x}-\underline{\tilde{X}})^{k-1}\\ &= -(k+1)(\underline{x}-\underline{\tilde{X}})^{k-1}.\end{aligned}$$

■

Proposition 4.4.5 *A polynomial of the form*

$$P_k(x_0,\underline{x})=\sum_{s=0}^{k}\binom{k}{s}a_s x_0^{k-s}(\underline{x}-\underline{\tilde{X}})^s, \tag{4.52}$$

where $a_s\in\mathbb{R}$, $a_0=1$, *and* $\underline{\tilde{X}}=\big(j_1x_1+j_2x_2+j_3x_3\big)\jmath$, *with* $\jmath^2=\big(j_1e_1+j_2e_2+j_3e_3\big)^2=-1$, *is holomorphic if and only if*

$$a_s=\frac{1}{2^s}\binom{s}{\lfloor\frac{s}{2}\rfloor}=c_s(2). \tag{4.53}$$

Proof.

We first prove that the form (4.53) of the coefficients a_s is a necessary condition for P_k being holomorphic. According to (4.52), the action of the operator ∂_0 on P_k gives

$$\begin{aligned}\partial_0P_k(x_0,\underline{x}) &= \sum_{s=0}^{k-1}(k-s)\binom{k}{s}a_s(k)x_0^{k-1-s}(\underline{x}-\underline{\tilde{X}})^s\\ &= \sum_{s=0}^{\lfloor\frac{k-1}{2}\rfloor}(k-2s)\binom{k}{2s}a_{2s}(k)x_0^{k-1-2s}(\underline{x}-\underline{\tilde{X}})^{2s}\\ &\qquad+\sum_{s=0}^{\lfloor\frac{k}{2}\rfloor-1}(k-1-2s)\binom{k}{2s+1}a_{2s+1}(k)x_0^{k-2-2s}(\underline{x}-\underline{\tilde{X}})^{2s+1}.\end{aligned} \tag{4.54}$$

Analogously, by means of Lemma 4.4.4 it follows that

$$\begin{aligned}\partial_{\underline{x}} P_k(x_0,\underline{x}) = &- \sum_{s=0}^{\lfloor\frac{k-1}{2}\rfloor} (2s+2)\binom{k}{2s+1} a_{2s+1}(k) x_0^{k-1-2s}(\underline{x}-\underline{\tilde{X}})^{2s} \\ &- \sum_{s=0}^{\lfloor\frac{k}{2}\rfloor-1} (2s+2)\binom{k}{2s+2} a_{2s+2}(k) x_0^{k-2-2s}(\underline{x}-\underline{\tilde{X}})^{2s+1}. \end{aligned} \tag{4.55}$$

Since P_k is holomorphic, $\partial_0 P_k(x_0,\underline{x}) = -\partial_{\underline{x}} P_k(x_0,\underline{x})$. Thus, after comparing the real and vector parts of (4.54) and (4.55) one obtains, after simplification,

$$a_s(k) = \begin{cases} \frac{2s+1}{2s+2} a_{s-1}(k), & s \quad \text{odd} \\ a_{s-1}(k), & s \quad \text{even}, \; s \geq 2 \end{cases}.$$

By observing that $a_0(k) = 1$, it follows immediately that $a_s(k) = c_s(2)$ (recall (2.10) for $n = 2$).

Next we prove that relation (4.53) implies the holomorphy of P_k. In this case, since

$$\begin{aligned}\partial_0 P_k(x_0,\underline{x}) &= \sum_{s=0}^{k-1} (k-s)\frac{1}{2^s}\binom{k}{s}\binom{s}{\lfloor\frac{s}{2}\rfloor} x_0^{k-1-s}\,(\underline{x}-\underline{\tilde{X}})^s \\ &= k\sum_{s=0}^{k-1} \frac{1}{2^s}\binom{k-1}{s}\binom{s}{\lfloor\frac{s}{2}\rfloor} x_0^{k-1-s}\,(\underline{x}-\underline{\tilde{X}})^s = kP_{k-1}, \end{aligned} \tag{4.56}$$

it is sufficient to prove that $\partial_{\underline{x}} P_k = -kP_{k-1}$. Thus, by means of Lemma 4.4.4 one has that

$$\begin{aligned}\partial_{\underline{x}} P_k(x_0,\underline{x}) &= \sum_{s=1}^{\lfloor\frac{k}{2}\rfloor} \binom{k}{2s}\binom{2s}{s}\frac{-2s}{2^{2s}}\, x_0^{k-2s}\,(\underline{x}-\underline{\tilde{X}})^{2s-1} \\ &\quad + \sum_{s=0}^{\lfloor\frac{k-1}{2}\rfloor} \binom{k}{2s+1}\binom{2s+1}{s}\frac{-(2s+2)}{2^{2s+1}}\, x_0^{k-1-2s}\,(\underline{x}-\underline{\tilde{X}})^{2s} \\ &= \sum_{s=0}^{\lfloor\frac{k-2}{2}\rfloor} \binom{k}{2s+2}\binom{2s+2}{s+1}\frac{-(s+1)}{2^{2s+1}}\, x_0^{k-2-2s}\,(\underline{x}-\underline{\tilde{X}})^{2s+1} \\ &\quad + \sum_{s=0}^{\lfloor\frac{k-1}{2}\rfloor} \binom{k}{2s+1}\binom{2s+1}{s}\frac{-(s+1)}{2^{2s}}\, x_0^{k-1-2s}\,(\underline{x}-\underline{\tilde{X}})^{2s}. \end{aligned}$$

Finally, since

$$(s+1)\binom{k}{2s+2}\binom{2s+2}{s+1} = k\binom{k-1}{2s+1}\binom{2s+1}{s}$$
$$(s+1)\binom{k}{2s+1}\binom{2s+1}{s} = k\binom{k-1}{2s}\binom{2s}{s}$$

it follows at once that $\partial_{\underline{x}} P_k = -kP_{k-1}$. ■

The characterization of all sequences of Appell polynomials with values in $\mathbb{R}^4$ is resumed in the following result.

Theorem 4.4.6 *For polynomials of the form*

$$P_k(x_0, \underline{x}) = \sum_{s=0}^{k} \binom{k}{s} d_s x_0^{k-s} (X_1 e_1 + X_2 e_2 + X_3 e_3)^s$$

there are exactly three different types of non-trivial Appell polynomials, namely

1. *The SAP* (2.11)

$$P_k(x_0, \underline{x}) = \mathcal{P}_k^n(x_0, \underline{x}) = \sum_{s=0}^{k} \binom{k}{s} c_s(3) x_0^{k-s} \underline{x}^s.$$

2. *The PCP* (4.31)

$$P_k(x_0, \underline{x}) = Z_{\vec{i}}^k = \sum_{s=0}^{k} \binom{k}{s} x_0^{k-s} (i_1 x_1 + i_2 x_2 + i_3 x_3)^s \hat{\imath}^s,$$

where $\hat{\imath}^2 = \big(i_1 e_1 + i_2 e_2 + i_3 e_3\big)^2 = -1.$

3. *The new Appell-type polynomials* (4.52)

$$P_k(x_0, \underline{x}) = \widetilde{P}_k(x_0, \underline{x}) = \sum_{s=0}^{k} \binom{k}{s} c_s(2) x_0^{k-s} (\underline{x} - \underline{\tilde{X}})^s,$$

where $\underline{\tilde{X}} = \big(i_1 x_1 + i_2 x_2 + i_3 x_3\big)\hat{\imath}$, *with* $\hat{\imath}^2 = \big(i_1 e_1 + i_2 e_2 + i_3 e_3\big)^2 = -1.$

Remark 4.4.7 Since

$$\mathrm{Vec}\left(\widetilde{P}_1(x_0, \underline{x})\right) = \frac{1}{2}(\underline{x} - \underline{\tilde{X}}),$$

it is possible to identify a connection of the vector part of these *new* Appell polynomials with the vector parts of the SAP and the PCP in $\mathbb{R}^4$, i.e. with $\mathrm{Vec}\left(\mathcal{P}_1^3(x)\right) = \frac{1}{3}\underline{x}$ and $\mathrm{Vec}\left(Z_{\vec{i}}\right) = x_{\hat{\imath}}\hat{\imath} = \underline{\tilde{X}}$, where $x_{\hat{\imath}} = i_1 x_1 + i_2 x_2 + i_3 x_3$ and $\hat{\imath} = i_1 e_1 + i_2 e_2 + i_3 e_3$ with $\hat{\imath}^2 = -1$. ♦

Comments to Chapter 4

We would like to add some comments on the role of generalized constants (see Remark 1.2.9) in the context of the Appell sequences considered in Theorem 4.4.6. The vector part of the first degree polynomial $P_1(x_0, \underline{x})$ in (4.38) can also be written as

$$d_1\underline{X} = d_1(a_1x_1e_1 + a_2x_2e_2 + a_3x_3e_3 + \lambda_1(x_1e_2 + x_2e_1) + \lambda_2(x_3e_1 + x_1e_3) + \lambda_3(x_2e_3 + x_3e_2)).$$

This means that $d_1\underline{X}$ can be decomposed into two essentially different components. The main term which is determined by the values of a_1, a_2 and a_3 has the form

$$d_1(a_1x_1e_1 + a_2x_2e_2 + a_3x_3e_3), \quad \text{where} \quad a_1 + a_2 + a_3 \neq 0,$$

while the second term

$$d_1[\lambda_1(x_1e_2 + x_2e_1) + \lambda_2(x_3e_1 + x_1e_3) + \lambda_3(x_2e_3 + x_3e_2)]$$

is nothing else than a linear combination of three generalized constants. The different cases of admissible $P_1(x_0, \underline{x})$ correspond to different choices of λ_k as functions of $a_k, i = 1, 2, 3$.

A complete characterization of generalized Appell sequences of polynomials of the form

$$P_k(x_0, \underline{x}) = \sum_{s=0}^{k} \binom{k}{s} d_s(n) x_0^{k-s} (X_1e_1 + \cdots + X_ne_n)^s,$$

for $n > 3$ was not carried out in this work. Indeed, when the dimension n increases, the complexity of the nonlinear system (4.42) obtained as a consequence of the holomorphy of P_2 should greatly enhance. Even so, Theorem 4.4.6 already shows that one should expect new types of Appell polynomials different from the SAP and the PCP for $n \geq 3$.

Chapter 5

Computational properties of pseudo-complex powers

5.1 Introduction

The construction of different bases of holomorphic polynomials in the framework of HFT is a topic which has been discussed by several authors and from different points of view ([10, 17, 21, 25, 29, 30, 48, 96, 102]).

Despite of the similar form of the Taylor series of a $\mathbb{C}-$valued holomorphic function of several complex variables with the Taylor series of a $\mathcal{C}\ell_{0,n}-$holomorphic function in terms of GP (see (1.37) and (1.38)), there are many differences between the complex and the hypercomplex settings which motivate the development of new structures and methods in HFT. Approximating a holomorphic function by its truncated Taylor series involves the computation of GP which, for large k, can become a difficult task.

The aim of this chapter is to construct explicitly bases of homogeneous $\mathcal{A}_2-$valued holomorphic polynomials formed by PCP. The reasons which led to our approach have their origins in the following observations:

1. For a fixed homogeneous degree k all PCP have the same structure, isomorphic to the structure of $z^k, z \in \mathbb{C}$;

2. The structure of PCP implies immediately reduced numerical costs for their construction;

3. Since PCP are powers of a normalized totally regular variable, they constitute auto-

matically a sequence of Appell polynomials.

In this chapter we concentrate essentially on a subject that has received far less attention: the numerical aspects of the construction of bases of holomorphic polynomials. The representation of the well known Fueter polynomial basis by a particular PCP-basis is subject to a detailed analysis for showing the numerical efficiency of the use of such PCP. In this context, a brief survey on numerical methods for inverting Vandermonde matrices is presented and a modified algorithm is proposed.

5.2 Bases of homogeneous polynomials constructed by pseudo-complex powers

We focus here on the construction of bases for $\mathcal{M}_k(\mathbb{H},\mathbb{H})$, the $\mathbb{H}$ space of homogeneous holomorphic $\mathbb{H}-$valued polynomials of degree k, formed by $k+1$ homogeneous holomorphic polynomials having the Appell property and whose structure is similar to the classical complex powers z^k.

We recall that in [25] and [29] the authors considered homogeneous holomorphic $\mathcal{A}_2$-valued polynomials closely related to the GP, namely

$$H^k_{(a_s,b_s)}(\vec{z}) = \big(a_s z_1 + b_s z_2\big)^k$$

$s = 0,\ldots,k$, and obtained, by imposing some conditions on the pairs of numbers (a_s, b_s), bases for $\mathcal{M}_k(\mathbb{H},\mathbb{H})$.

Theorem 5.2.1 ([25], Theorem 2.4.1) *For a fixed $k \in \mathbb{N}_0$ and given a set of $k+1$ pairs of real numbers (a_s, b_s), such that $a_s b_r - a_r b_s \neq 0,\ s \neq r,\ s,r = 0,1,\ldots,k,$ the set*

$$\big\{H^k_{(a_0,b_0)}, H^k_{(a_1,b_1)}, \ldots, H^k_{(a_k,b_k)}\big\} \tag{5.1}$$

forms a basis for $\mathcal{M}_k(\mathbb{H},\mathbb{H})$.

Associating each pair (a_s, b_s) to a vector in $\mathbb{R}^2$ we can observe that the required conditions $a_s b_r - a_r b_s \neq 0,\ s \neq r$, correspond to the choice of $k+1$ vectors in $\mathbb{R}^2$ pairwise non-collinear.

Besides their connection with the Fueter variables, all the elements of (5.1) share the same structure and their hypercomplex derivative is

$$\partial H^k_{(a_s,b_s)}(\vec{z}) = -k\big(a_s z_1 + b_s z_2\big)^{k-1}(a_s e_1 + b_s e_2) = -k H^{k-1}_{(a_s,b_s)}(\vec{z})(a_s e_1 + b_s e_2).$$

Based on this, in [30] the authors observed that the multiplication of $H^k_{(a_s,b_s)}(\vec{z})$ by the constant

$$\Big(\frac{a_s e_1 + b_s e_2}{a_s^2+b_s^2}\Big)^k$$

leads to a basis of Appell polynomials for $\mathcal{M}_k(\mathbb{H},\mathbb{H})$. Observe that this corresponds to consider the polynomials

$$H^k_{(a_s,b_s)}(\vec{z})\Big(\frac{a_s e_1 + b_s e_2}{a_s^2+b_s^2}\Big)^k = \Big(\frac{a_s z_1 + b_s z_2}{\sqrt{a_s^2+b_s^2}}\Big)^k\Big(\frac{a_s e_1 + b_s e_2}{\sqrt{a_s^2+b_s^2}}\Big)^k. \tag{5.2}$$

Thus, by defining

$$i_{s1} = \frac{a_s}{\sqrt{a_s^2+b_s^2}}, \quad i_{s2} = \frac{b_s}{\sqrt{a_s^2+b_s^2}}, \quad \hat{\imath}_s = i_{s1}e_1 + i_{s2}e_2, \tag{5.3}$$

it follows at once that $\hat{\imath}_s^2 = -1$ and we recognize in (5.2) the PCP of order k associated with the pure unit vector $\hat{\imath}_s$, introduced in the previous chapter (cf. (4.32) for $n=2$), i.e.

$$Z^k_{\vec{\imath}_s} = (z_1 i_{s1} + z_2 i_{s2})^k (i_{s1}e_1 + i_{s2}e_2)^k. \tag{5.4}$$

Therefore, relation (5.2) can be rewritten as

$$H^k_{\vec{\imath}_s}(\vec{z})\hat{\imath}_s^k = Z^k_{\vec{\imath}_s},$$

where $\hat{\imath}_s$ is the pure unit vector in (5.3), and the following result is straightforward.

Corollary 5.2.2 *The set of polynomials of the form* (5.4)

$$\big\{Z^k_{\vec{\imath}_0}, Z^k_{\vec{\imath}_1}, \ldots, Z^k_{\vec{\imath}_k}\big\} \tag{5.5}$$

forms a basis for $\mathcal{M}_k(\mathbb{H},\mathbb{H})$, provided that the $k+1$ unit vectors $\vec{\imath}_s = (i_{s1}, i_{s2})$, $s = 0,\ldots,k$, are pairwise non-collinear.

Taking into account the role played by the unit vectors in the last result, it seems natural to introduce the following definition closely related to the construction of bases of PCP proposed along this chapter.

Definition 5.2.3 *Let $Z^k_{\vec{\imath}_s}$ be the PCP associated with $\vec{\imath}_s, s = 0,\ldots,k$. We say that the set Π_k of $k+1$ unit vectors $\vec{\imath}_s$ is a* ***parameter set of order*** *k* ***associated with*** *$\{Z^k_{\vec{\imath}_0}, Z^k_{\vec{\imath}_1}, \ldots, Z^k_{\vec{\imath}_k}\}$ if the vectors are pairwise non-collinear. In other words,*

$$\Pi_k = \{\vec{\imath}_0, \vec{\imath}_1, \ldots, \vec{\imath}_k\} = \{(i_{01}, i_{02}), (i_{11}, i_{12}), \ldots, (i_{k1}, i_{k2})\} \tag{5.6}$$

where $\vec{\imath}_s = (i_{s1}, i_{s2})$ is such that $i_{s1}^2 + i_{s2}^2 = 1$, $s = 0,1,\ldots,k$ and, for $\vec{\imath}_r \in \Pi_k$, with $r \neq s$, the relation $i_{s1}i_{r2} - i_{r1}i_{s2} \neq 0$ holds, with $r,s = 0,\ldots,k$.

Our aim here is to compare the GP basis (1.33) and bases of PCP of the form (5.5) from several points of view. For such purpose we observe the following:

1. The elements of (5.5) obey all the same principle structure of the complex powers z^k, which suggest the idea of using complex arithmetic for their evaluation. We will see in Section 5.4 that this is the key idea for showing the numerical efficiency of PCP. In turn, each term of (1.33) is defined in terms of the symmetric product and the only polynomials of degree k which can be identified with $z^k \in \mathbb{C}$ are z_1^k and z_2^k. This means that the remaining polynomials require quaternionic arithmetic for their evaluation.

2. Since the PCP of first degree are TRV we have that each element of (5.5) verifies
$$Z_{\vec{\imath}_s}^k = \big(Z_{\vec{\imath}_s}\big)^k = \big(Z_{\vec{\imath}_s}\big)^{k-1} Z_{\vec{\imath}_s}. \tag{5.7}$$
As far as the GP are concerned, it is possible to relate GP of consecutive degree by means of the recursive formula (1.28) (or (1.29)) which, for $n = 2$ reads as follows
$$z_1^{k-s} \times z_2^s = \frac{1}{k}\Big((k-s) z_1 z_1^{k-1} \times z_2^s + s\, z_2 z_1^{k-s} \times z_2^{s-1}\Big), \quad s = 0, \ldots, k. \tag{5.8}$$
It seems intuitive that the use of the recursion formula (5.8) will be computationally more expensive than (5.7). This will be discussed in detail later on in this chapter.

3. Regarding the hypercomplex derivative, since the PCP form an Appell system, it follows immediately that
$$\partial Z_{\vec{\imath}}^k = k Z_{\vec{\imath}}^{k-1}.$$
On the other hand, according to [77], the hypercomplex derivative of a given GP has the form
$$\partial\big(z_1^{k-s} \times z_2^s\big) = -(k-s)\big(z_1^{k-s-1} \times z_2^s\big)e_1 - s\big(z_1^{k-s} \times z_2^{s-1}\big)e_2,$$
which means that the derivative of each GP is a linear combination of two polynomials of degree $k-1$ and not a multiple of a polynomial of the same system.

Having all this in consideration, in what follows we will see how to obtain the explicit expression of each GP $z_1^{k-s} \times z_2^s$ in terms of a basis of PCP $\{Z_{\vec{\imath}_s}^k\}_{s=0}^k$ for a chosen parameter set. This will allow to express the series expansion of an $\mathcal{A}_2$-valued holomorphic function in terms of PCP.

By recalling the notion of a parameter set (5.6) and the expression (5.4) of each PCP one obtains

$$\begin{cases} Z_0^k = (z_1 i_{01} + z_2 i_{02})^k (i_{01} e_1 + i_{02} e_2)^k \\ Z_1^k = (z_1 i_{11} + z_2 i_{12})^k (i_{11} e_1 + i_{12} e_2)^k \\ \vdots \\ Z_{k-1}^k = (z_1 i_{k-1,1} + z_2 i_{k-1,2})^k (i_{k-1,1} e_1 + i_{k-1,2} e_2)^k \\ Z_k^k = (z_1 i_{k1} + z_2 i_{k2})^k (i_{k1} e_1 + i_{k2} e_2)^k \end{cases}, \tag{5.9}$$

where Z_s^k is the shorthand notation for $Z_{\vec{\imath}_s}^k$. Recalling the binomial formula (1.31), we can write (5.9) as

$$\mathcal{Z}^k = \boldsymbol{A_k} \mathcal{G}^k \tag{5.10}$$

where $\mathcal{Z}^k$ and $\mathcal{G}^k$ are $(k+1)$ column vectors containing in each row the PCP and the GP of degree k, i.e.

$$\mathcal{Z}^k = \begin{pmatrix} Z_0^k \\ Z_1^k \\ \vdots \\ Z_k^k \end{pmatrix}, \qquad \mathcal{G}^k = \begin{pmatrix} z_1^k \\ z_1^{k-1} \times z_2 \\ \vdots \\ z_2^k \end{pmatrix} \tag{5.11}$$

and

$$\boldsymbol{A_k} = \begin{pmatrix} \binom{k}{0} i_{01}^k \hat{\imath}_0^k & \binom{k}{1} i_{01}^{k-1} i_{02} \hat{\imath}_0^k & \cdots & \binom{k}{k} i_{02}^k \hat{\imath}_0^k \\ \binom{k}{0} i_{11}^k \hat{\imath}_1^k & \binom{k}{1} i_{11}^{k-1} i_{12} \hat{\imath}_1^k & \cdots & \binom{k}{k} i_{12}^k \hat{\imath}_1^k \\ \vdots & \vdots & \ddots & \vdots \\ \binom{k}{0} i_{k1}^k \hat{\imath}_k^k & \binom{k}{1} i_{k1}^{k-1} i_{k2} \hat{\imath}_k^k & \cdots & \binom{k}{k} i_{k2}^k \hat{\imath}_k^k \end{pmatrix}. \tag{5.12}$$

The matrix $\boldsymbol{A_k}$ admits a factorization which involves a Vandermonde matrix of order $k+1$ or order k, depending on the chosen parameter set Π_k. Before proving this we observe the following:

1. Since the elements $\vec{\imath}_s = (i_{s1}, i_{s2})$ of the parameter set Π_k are unitary, if one of the vector components vanishes then the other one equals ± 1.

2. Due to the non-collinear property there is at most one vector in Π_k with a first null component. In such case we assume, for simplicity, that the null component is i_{01} and, in this case we consider $i_{02} = 1$. The treatment of the case where the unique first null component is not i_{01} or the second component is -1 is completely analogous.

Based on this we will analyze the factorization of $\boldsymbol{A_k}$ in the following two situations:

Case 1: The parameter set Π_k is of the form

$$\Pi_k = \{(i_{01}, i_{02}), (i_{11}, i_{12}), \ldots, (i_{k1}, i_{k2})\}, \quad i_{s1} \neq 0, \quad i_{s2} \neq 0, \quad s = 0, \ldots, k. \tag{5.13}$$

In this case it follows that

$$\boldsymbol{A_k} = B_k V(X_{k+1}) C_k, \tag{5.14}$$

where B_k and C_k are the diagonal matrices of order $k+1$

$$B_k = \operatorname{diag}\big(\hat{\imath}_0^k i_{01}^k, \hat{\imath}_1^k i_{11}^k, \ldots, \hat{\imath}_k^k i_{k1}^k\big), \tag{5.15}$$

$$C_k = \operatorname{diag}\Big(\tbinom{k}{0}, \tbinom{k}{1}, \ldots, \tbinom{k}{k}\Big) \tag{5.16}$$

and $V(X_{k+1})$ denotes the Vandermonde matrix of order $(k+1)$ associated with the set of nodes

$$X_{k+1} = \left\{ \frac{i_{02}}{i_{01}}, \frac{i_{12}}{i_{11}}, \ldots, \frac{i_{k2}}{i_{k1}} \right\}, \tag{5.17}$$

i.e.

$$V(X_{k+1}) = \begin{pmatrix} 1 & \frac{i_{02}}{i_{01}} & \cdots & \left(\frac{i_{02}}{i_{01}}\right)^k \\ 1 & \frac{i_{12}}{i_{11}} & \cdots & \left(\frac{i_{12}}{i_{11}}\right)^k \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \frac{i_{k2}}{i_{k1}} & \cdots & \left(\frac{i_{k2}}{i_{k1}}\right)^k \end{pmatrix}.$$

Case 2: The parameter set Π_k is of the form

$$\Pi_k = \{(0, 1), (i_{11}, i_{12}), \ldots, (i_{k1}, i_{k2})\}, \quad i_{s1} \neq 0, \quad i_{s2} \neq 0, \quad s = 1, \ldots, k. \tag{5.18}$$

In this case the matrix $\boldsymbol{A_k}$ admits the following factorization

$$\boldsymbol{A_k} = \widetilde{B}_k H(Y_k) \widetilde{V}(Y_k) C_k \tag{5.19}$$

where $\widetilde{B}_k$ is the diagonal matrix of order $k+1$ given by

$$\widetilde{B}_k = \operatorname{diag}\big(\hat{\imath}_0^k, \hat{\imath}_1^k i_{11}^k, \ldots, \hat{\imath}_k^k i_{k1}^k\big), \tag{5.20}$$

C_k is the matrix (5.16) and $H(Y_k)$ and $\widetilde{V}(Y_k)$ are the block matrices

$$H(Y_k) = \left(\begin{array}{c|c} \mathbf{0}_k & 1 \\ \hline I_k & \begin{matrix} \left(\frac{i_{12}}{i_{11}}\right)^k \\ \vdots \\ \left(\frac{i_{k2}}{i_{k1}}\right)^k \end{matrix} \end{array}\right), \qquad \widetilde{V}(Y_k) = \left(\begin{array}{c|c} V(Y_k) & \mathbf{0}_k^T \\ \hline \mathbf{0}_k & 1 \end{array}\right). \tag{5.21}$$

Here, as usual, I_k denotes the identity of order k, $\mathbf{0}_k$ is the k null row vector and Y_k is the set obtained from X_{k+1} by removing the first node, i.e.

$$Y_k = \left\{\frac{i_{12}}{i_{11}}, \frac{i_{22}}{i_{21}}, \ldots, \frac{i_{k2}}{i_{k1}}\right\}. \tag{5.22}$$

According to (5.10), it is possible to express each GP in terms of a basis of PCP provided that the matrices in (5.14) or (5.19) are invertible. It follows immediately that all the matrices $B_k, \widetilde{B}_k, C_k$ and $H(Y_k)$ related to Cases 1 and 2 have non-null determinant. Thus, it remains to analyze the invertibility of the Vandermonde matrices $V(X_{k+1})$ and $V(Y_k)$.

It is well known that the determinant of a Vandermonde matrix with nodes $x_1, \ldots, x_k$ is $\prod_{1\le m<n\le k}(x_n - x_m)$. Therefore the determinants of the aforementioned Vandermonde matrices are

$$|V(X_{k+1})| = \prod_{0\le m<n\le k}\left(\frac{i_{m2}}{i_{m1}} - \frac{i_{n2}}{i_{n1}}\right) \qquad \text{and} \qquad |V(Y_k)| = \prod_{1\le m<n\le k}\left(\frac{i_{m2}}{i_{m1}} - \frac{i_{n2}}{i_{n1}}\right).$$

Based on the non-collinear property of the unit vectors $\vec{\imath}_s \in \Pi_k$, it follows immediately that $|V(X_{k+1})| \neq 0$ and $|V(Y_k)| \neq 0$ and consequently $V(X_{k+1})$ and $\widetilde{V}(Y_k)$ are both invertible.

The problem of inverting Vandermonde matrices arises in several research areas and it is for its own sake an interesting subject which is still object of investigation. The inversion of $\boldsymbol{A}_k$ in (5.10) was the motivation and the starting point for considering efficient algorithms to reach this goal. During this task we managed to derive general results which, in our opinion, represent a new contribution to the area. For this reason we include in this work a section totally devoted to the problem of inverting Vandermonde matrices.

5.3 On the inversion of Vandermonde matrices

Vandermonde matrices $V(X_k)$, with $X_k = \{x_1, \ldots, x_k\}$, or simply $V(x_1, \ldots, x_k)$, have been widely used in several problems of applied and numerical analysis. These matrices are mostly known by their applications in Lagrange interpolating problems, namely in the problem of approximating a real valued function by a polynomial of degree $k-1$ that interpolates the function at the k points or nodes x_j, $j = 1, \ldots, k$. The coefficients of such polynomial are uniquely determined by means of the inverse of the correspondent matrix $V(x_1, \ldots, x_k)$, provided that the nodes are all distinct.

Vandermonde matrices also appear associated with systems of differential equations, control, optimization and approximation problems. In [72] one can find many other references on this topic.

One of the drawbacks of Vandermonde matrices with real nodes is the fact that they are known to be often quite ill-conditioned[1] and standard numerically stable methods in general fail to accurately compute the entries of their inverses (cf. [52]). For this reason the inversion of a Vandermonde matrix has received much attention, particularly in the search for fast and accurate algorithms.

In the end of the decade of 1940 the work [61] presented an algorithm for obtaining the coefficients of a polynomial of degree $k-1$ which fits k given points. Even without mentioning Vandermonde matrix as well as its inverse, it already reveals the importance of this topic. Later on, in 1966, L. Turner presented in [137] the UL-decomposition of the inverse of the Vandermonde matrix $V(X_k)$. In this case the generic entries of the two triangular matrices are explicitly determined. In the same year, J. Traub gave in [136] an algorithm[2] which computes the inverse of $V(X_k)$ in $6k^2$ floating point operations (flops). Unfortunately, this algorithm is widely regarded as being numerically unstable. Recently in [52] it was proposed an algorithm which is competitive in terms of both numerical accuracy and computational effort.

We follow here the notations of this work in order to present the referred algorithm, starting by first defining the following well-known elementary functions.

Definition 5.3.1 *Let k, s be non-negative integers. The elementary symmetric polynomial $\sigma(k,s)$ is defined as*

$$\sigma(k,s) = \sum_{1 \leq j_1 < \cdots < j_s \leq k} x_{j_1} \ldots x_{j_s}. \tag{5.23}$$

[1] A problem is said to be *ill-conditioned* if a small error in the data or in subsequent calculation results in much larger errors in the answers. The problem of measuring the effect of errors when computing the inverse of a matrix A is, in general, associated with the *condition number* of A. This number is given by $\operatorname{cond} A = ||A||\,||A^{-1}||$ and is a measure of the (relative) errors in the inverse of A when the latter is perturbed by an arbitrary infinitesimally small matrix.

[2] If the structure of $V(X_k)$ is ignored, then usually $O(k^3)$ flops are required to compute $V(X_k)^{-1}$.

We note that $\sigma(k,s)$ corresponds to the sum of all products of s distinct nodes chosen from $X_k = \{x_1, \dots, x_k\}$. Moreover:

Property 5.3.2 *The elementary symmetric polynomial $\sigma(k,s)$ can be defined recursively as*

$$\begin{aligned} &\sigma(k,0) = 1, \\ &\sigma(k,s) = 0, \quad \textit{for } s > k \textit{ or } s < 0, \\ &\sigma(k,s) = \sigma(k-1,s) + x_k\sigma(k-1,s-1). \end{aligned} \tag{5.24}$$

Property 5.3.3 *The generating function of $\sigma(k,s)$ is given by*

$$S_k(x) = \prod_{j=1}^{k}(x - x_j) = \sum_{r=0}^{k}(-1)^{k+r}\sigma(k,k-r)x^r. \tag{5.25}$$

Last property gives immediately that $S_k(x_j) = 0, j = 1, \dots, k$, which is equivalent to write

$$\sum_{r=0}^{k}(-1)^r\sigma(k,k-r)x_m^r = 0 \quad m = 1, \dots, k. \tag{5.26}$$

Definition 5.3.4 *The **barycentric weights**[3] associated with X_k are defined by*

$$\begin{aligned} &\Phi(2,1) = \Phi(2,2) = \frac{1}{x_2 - x_1} \\ &\Phi(i,i) = \prod_{r=1}^{i-1}\frac{1}{x_i - x_r}, \\ &\Phi(i,j) = \frac{\Phi(i-1,j)}{x_i - x_j}, \quad i = 1, \dots, k, \ \ j = 1, \dots, i. \end{aligned} \tag{5.27}$$

We are now in position to present the main result of [52] which gives the explicit formula for the inverse of a Vandermonde matrix.

Theorem 5.3.5 ([52], Theorem 1) *The inverse of the Vandermonde matrix $V(X_k)$, here denoted by $W(X_k)$, has as its generic element*

$$w_{ij} = (-1)^{i+j}\Phi(k,j)\sum_{r=0}^{k-i}(-1)^r x_j^r\sigma(k,k-i-r), \quad i,j = 1,2,\dots,k. \tag{5.28}$$

[3] A manipulation of the Lagrange polynomial through the formulas of barycentric interpolation, which involves the barycentric weights, leads to a faster and stable variant of the Lagrange interpolation (see [14] for details on this topic).

To prove this result the authors used the expressions (5.23), (5.26) and (5.27) to conclude that $V(X_k)W(X_k)$ is the identity matrix I_k.

We note that if one of the nodes in X_k, say x_r, is zero then

$$w_{1r} = 1 \qquad \text{and} \qquad w_{1j} = 0, \;\; j \neq r. \tag{5.29}$$

The following examples show the Vandermonde matrices and their inverses for the cases of $k = 2$ and $k = 3$.

Example 5.3.1 If $X_2 = \{x_1, x_2\}$, then

$$V(X_2) = \begin{pmatrix} 1 & x_1 \\ 1 & x_2 \end{pmatrix},$$

$$\sigma(2,0) = 1, \quad \sigma(2,1) = x_1 + x_2, \quad \sigma(2,2) = x_1 x_2,$$

$$\Phi(2,1) = \Phi(2,2) = \frac{1}{x_2 - x_1}$$

and

$$W(X_2) = \begin{pmatrix} \frac{x_2}{x_2 - x_1} & -\frac{x_1}{x_2 - x_1} \\ -\frac{1}{x_2 - x_1} & \frac{1}{x_2 - x_1} \end{pmatrix}.$$

Example 5.3.2 If $X_3 = \{x_1, x_2, x_3\}$, then

$$V(X_3) = \begin{pmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{pmatrix},$$

$$\sigma(3,0) = 1, \quad \sigma(3,1) = x_1 + x_2 + x_3, \quad \sigma(3,2) = x_1 x_2 + x_2 x_3 + x_3 x_1, \quad \sigma(3,3) = x_1 x_2 x_3,$$

$$\Phi(3,1) = \frac{1}{(x_2 - x_1)(x_3 - x_1)}, \; \Phi(3,2) = \frac{1}{(x_2 - x_1)(x_3 - x_2)}, \; \Phi(3,3) = \frac{1}{(x_3 - x_1)(x_3 - x_2)}$$

and

$$W(X_3) = \begin{pmatrix} \frac{x_2 x_3}{(x_2 - x_1)(x_3 - x_1)} & \frac{x_1 x_3}{(x_1 - x_2)(x_3 - x_2)} & \frac{x_1 x_2}{(x_2 - x_3)(x_1 - x_3)} \\ \frac{x_2 + x_3}{(x_1 - x_2)(x_3 - x_1)} & \frac{x_1 + x_3}{(x_1 - x_2)(x_2 - x_3)} & \frac{x_1 + x_2}{(x_2 - x_3)(x_3 - x_1)} \\ \frac{1}{(x_2 - x_1)(x_3 - x_1)} & \frac{1}{(x_1 - x_2)(x_3 - x_2)} & \frac{1}{(x_3 - x_2)(x_3 - x_1)} \end{pmatrix}.$$

As already mentioned, a different way for obtaining the inverse of the Vandermonde matrix goes back to the article [137] of Turner. In this work the author obtained a tacit "UL" factorization of $W(X_k)$, where U is a unit upper triangular matrix with generic entries given by

$$u_{ij} = u_{i-1,j-1} - u_{i,j-1}x_{j-1}, \quad i = 1, \dots, k; \; j = i+1, \dots, k, \tag{5.30}$$

and L is a lower triangular matrix with generic entries given by

$$\begin{aligned} l_{ij} &= (-1)^{i+j}\Phi(i,j), \quad i = 1, \dots, k; \; j = 1, \dots, i, \\ l_{ij} &= 0 \quad \text{otherwise.} \end{aligned} \tag{5.31}$$

We observe that the elements of U can also be written as

$$u_{ij} = (-1)^{j-i}\sigma(j-1, j-i), \quad j = i+1, \dots, k. \tag{5.32}$$

In fact, recalling Property 5.3.2 we have that

$$\begin{aligned} u_{ij} &= (-1)^{j-i}\big(\sigma(j-2, j-i) + x_{j-1}\sigma(j-2, j-i-1)\big) \\ &= u_{i-1,j-1} - x_{j-1}u_{i,j-1}. \end{aligned}$$

For example, if $X_3 = \{x_1, x_2, x_3\}$, then

$$U = \begin{pmatrix} 1 & -x_1 & x_1x_2 \\ 0 & 1 & -(x_1+x_2) \\ 0 & 0 & 1 \end{pmatrix},$$

$$L = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{x_1-x_2} & \frac{1}{x_2-x_1} & 0 \\ \frac{1}{(x_1-x_2)(x_1-x_3)} & \frac{1}{(x_2-x_1)(x_2-x_3)} & \frac{1}{(x_3-x_1)(x_3-x_2)} \end{pmatrix}.$$

In the literature one can find several works related to *fast* algorithms for inverting the Vandermonde matrix. We may say that the primary goal of these papers is the comparison of the proposed algorithms with others in terms of numerical stability and accuracy (see [52], [72] and [73]). The question of obtaining recursive relations for the generic entry of the inverse of the Vandermonde matrix $V(X_k)$, $X_k = \{x_1, \dots, x_k\}$, without using block decomposition was not, to the best of our knowledge, a topic of research. In many practical applications the problem of adding (or removing) a new node to X_k occurs frequently and therefore it seems natural to look for a way of obtaining the element of the inverse of the new Vandermonde matrix without computing all the entries again. In other words, we focus here on the problem of calculating the inverse of a Vandermonde matrix associated with X_k, once the inverse of the Vandermonde matrix associated with $X_{k-1} \subset X_k$ is known. In the present context, this procedure can be of special interest if we want to obtain a series expansion similar to (1.37) (for $n = 2$) but involving PCP.

We distinguish the generic entries of the matrices $W(X_{k-1})$ and $W(X_k)$ by using w_{ij}^{k-1} and w_{ij}^{k}, respectively. The following result expresses the relation between these generic elements.

Lemma 5.3.6 *For $i = 1, \ldots, k$ and $j = 1, \ldots, k-1$, the following relation holds*

$$w_{ij}^{k} = \frac{w_{i-1,j}^{k-1} - x_k w_{ij}^{k-1}}{x_j - x_k}, \tag{5.33}$$

where $w_{0j}^{k} = 0$.

Proof. By recalling relations (5.24) and (5.27) we obtain

$$\begin{aligned}
w_{ij}^{k} &= (-1)^{k+i}\Phi(k,j)\sum_{r=0}^{k-i}(-1)^r x_j^r \sigma(k, k-i-r) \\
&= (-1)^{k+i}\frac{\Phi(k-1,j)}{x_j - x_k}\sum_{r=0}^{k-i}(-1)^r x_j^r \sigma(k-1, k-i-r) \\
&\qquad - (-1)^{k-1-i}\frac{x_k}{x_j - x_k}\Phi(k-1,j)\sum_{r=0}^{k-i}(-1)^r x_j^r \sigma(k-1, k-1-i-r) \\
&= \frac{w_{i-1,j}^{k-1} - x_k w_{ij}^{k-1}}{x_j - x_k}.
\end{aligned}$$

We note that for $i = 1$ the sum $\sum_{r=0}^{k-1}(-1)^r x_j^r \sigma(k-1, k-1-r)$ vanishes, due to relation (5.26). Therefore $w_{0j}^{k} = 0$. ∎

Theorem 5.3.7 *The inverse $W(X_k)$ of the Vandermonde matrix $V(X_k)$ can be written as*

$$W(X_k) = \left(\begin{array}{c|c} M_{k-1}W(X_{k-1})D_{k-1} & \boldsymbol{\Sigma}_{k-1} \\ \hline \boldsymbol{\Phi}_{k-1} & \Phi(k,k) \end{array}\right), \tag{5.34}$$

where M_{k-1} is the lower bidiagonal matrix whose elements are given by

$$m_{ij} = \begin{cases} x_k & i = j \\ -1 & i = j+1 \end{cases} \qquad i, j = 1, \ldots, k-1, \tag{5.35}$$

D_{k-1} is the diagonal matrix

$$D_{k-1} = \operatorname{diag}\Big(-\frac{1}{x_1 - x_k}, -\frac{1}{x_2 - x_k}, \dots, -\frac{1}{x_{k-1} - x_k} \Big), \tag{5.36}$$

$\mathbf{\Sigma}_{k-1}$ denotes the $(k-1)-$column vector and $\mathbf{\Phi}_{k-1}$ denotes the $(k-1)-$row vector whose elements are given by

$$(-1)^{k+i}\sigma(k-1,k-i)\Phi(k,k) \quad \textit{and} \quad (-1)^{k+j}\Phi(k,j), \quad i,j = 1,\dots,k-1, \tag{5.37}$$

respectively.

Proof. We want to prove that the elements of the matrix in the right-hand side of (5.34), which we denote by $\mathring{\mathrm{w}}_{ij}^k$, coincide with the elements w_{ij}^k of $W(X_k)$.

Since

$$\mathring{\mathrm{w}}_{ij}^k = \begin{cases} \dfrac{x_k w_{1j}^{k-1}}{x_k - x_j} & i = 1,\ j = 1,\dots,k-1 \\ \dfrac{x_k w_{ij}^{k-1} - w_{i-1,j}^{k-1}}{x_k - x_j} & i = 2,\dots,k-1,\ j = 1,\dots,k-1 \\ (-1)^{k+j}\Phi(k,j) & i = k,\ j = 1,\dots,k \\ (-1)^{k+i}\sigma(k-1,k-i)\Phi(k,k) & i = 1,\dots,k-1,\ j = k. \end{cases} \tag{5.38}$$

then, by means of Lemma 5.3.6 we conclude that $\mathring{\mathrm{w}}_{ij} = w_{ij}^k$ for $i,j = 1,\dots,k-1$.

Last row of $W(X_k)$ follows directly from the expression of w_{kj}^k in (5.28). Finally we consider $\mathbf{\Sigma}_{k-1}$. From (5.28), it follows using (5.24) that

$$w_{ik}^k = (-1)^{k+i}\Phi(k,k)\sum_{r=0}^{k-i}(-1)^r x_k^r\big(\sigma(k-1,k-i-r) + x_k\sigma(k-1,k-1-i-r)\big),$$

which is equivalent to

$$\begin{aligned} w_{ik}^k &= (-1)^{k+i}\Phi(k,k)\sigma(k-1,k-i) + (-1)^{k+i}\Phi(k,k)\sum_{r=1}^{k-i}(-1)^r x_k^r\sigma(k-1,k-i-r) \\ &\quad + (-1)^{k+i}\Phi(k,k)\sum_{r=0}^{k-i-1}(-1)^r x_k^{r+1}\sigma(k-1,k-1-i-r) \\ &= (-1)^{k+i}\Phi(k,k)\sigma(k-1,k-i) = \mathring{\mathrm{w}}_{i,k}^k. \end{aligned}$$

■

5.4 Generalized powers versus pseudo-complex powers

In this section we return to the problem of expressing each GP in terms of a basis of PCP for a chosen parameter set. Throughout this section we consider the PCP $Z_s^k = Z_{\vec{\imath}_s}^k$, $s = 0, \ldots, k$ associated with a parameter set Π_k of the form (5.13) or (5.18). In fact, using the tools introduced in the previous section, the problem of inverting $\boldsymbol{A_k}$ in (5.10) can now be solved.

Theorem 5.4.1 *Let Π_k be a parameter set of order k associated with $\{Z_s^k\}_{s=0}^k$. Each GP of degree k can be written as a linear combination of PCP as*

$$z_1^{k-s} \times z_2^s = \sum_{t=1}^{k+1} \boldsymbol{\alpha}_{s+1,t} Z_{t-1}^k, \quad s = 0, \ldots, k, \tag{5.39}$$

where the coefficients $\boldsymbol{\alpha}_{mn}$, depending on the form of Π_k, can be obtained as follows:

Case 1: *If Π_k is of the form (5.13), then*

$$\boldsymbol{\alpha}_{mn} = \binom{k}{m-1}^{-1} (i_{n-1,1}\hat{\imath}_{n-1})^{-k}\, w_{mn}^{k+1}, \tag{5.40}$$

where w_{mn}^{k+1} denotes the generic entry of $W(X_{k+1}) = V^{-1}(X_{k+1})$ and X_{k+1} is the set (5.17).

Case 2: *If Π_k is of the form (5.18), then*

$$\boldsymbol{\alpha}_{mn} = \begin{cases} -\hat{\imath}_0^{-k} \displaystyle\sum_{t=1}^{k} \binom{k}{m-1}^{-1} \widetilde{w}_{mt}^k \left(\frac{i_{t2}}{i_{t1}}\right)^k, & n = 1,\ m = 1, \ldots, k \\ 0, & m = k+1,\ n = 2, \ldots, k+1 \\ 1, & m = k+1,\ n = 1 \\ \binom{k}{m-1}^{-1} (i_{n-1,1}\hat{\imath}_{n-1})^{-k}\, \widetilde{w}_{m,n-1}^k, & otherwise \end{cases}, \tag{5.41}$$

where $\widetilde{w}_{mn}^k$ denotes the generic entry of $\widetilde{W}(Y_k) = \widetilde{V}^{-1}(Y_k)$ and Y_k is the set (5.22).

Proof. For the first case, by recalling (5.10) and (5.14) we can write

$$\mathcal{G}^k = C_k^{-1} W(X_{k+1}) B_k^{-1} \mathcal{Z}^k.$$

Since B_k and C_k are the diagonal matrices (5.15) and (5.16), respectively, it follows that the generic entry of $\boldsymbol{A_k^{-1}}$ is

$$\boldsymbol{\alpha}_{mn} = \binom{k}{m-1}^{-1} (i_{n-1,1}\hat{\imath}_{n-1})^{-k}\, w_{mn}^{k+1}.$$

Concerning the Case 2, from (5.10) and (5.19) we have

$$\mathcal{G}^k = C_k^{-1}\widetilde{W}(Y_k)H^{-1}(Y_k)\widetilde{B}_k^{-1}\mathcal{Z}^k.$$

In such case, it can be checked easily that the inverses of the block matrices in (5.21), i.e. $H(Y_k)$ and $\widetilde{V}(Y_k)$ are given respectively by

$$H^{-1}(Y_k) = \left(\begin{array}{c|c} -\left(\frac{i_{12}}{i_{11}}\right)^k & \\ -\left(\frac{i_{22}}{i_{21}}\right)^k & \\ \vdots & I_k \\ -\left(\frac{i_{k2}}{i_{k1}}\right)^k & \\ \hline 1 & \mathbf{0}_k \end{array}\right), \qquad \widetilde{W}(Y_k) = \left(\begin{array}{c|c} W(Y_k) & \mathbf{0}_k^T \\ \hline \mathbf{0}_k & 1 \end{array}\right) \tag{5.42}$$

where $W(Y_k) = V^{-1}(Y_k)$. Moreover, since $\widetilde{B}_k$ and C_k are the diagonal matrices (5.20) and (5.16), respectively, then the final form of the generic entries $\boldsymbol{\alpha}_{mn}$, i.e. (5.41), follows now at once. ■

Remark 5.4.2 We point out that the coefficients $\boldsymbol{\alpha}_{s+1,t}$ in (5.39), depending on the parity of k, are real (if k is even) or pure vectors (if k is odd). This means that in both expressions (5.40) and (5.41) the paravector $\hat{\imath}_s^{-k}$ can be written as

$$\hat{\imath}_s^{-k} = (\hat{\imath}_s^{-1})^k = (-\hat{\imath}_s)^k = \begin{cases} (-1)^{\frac{k}{2}} & k \quad \text{even} \\ (-1)^{\frac{k+1}{2}}\,\hat{\imath}_s & k \quad \text{odd} \end{cases}.$$

♦

To illustrate the applicability of Theorem 5.4.1, consider the particular cases of $k = 1$ and $k = 2$.

Example 5.4.1 Let $\Pi_1 = \{\vec{\imath}_0, \vec{\imath}_1\} = \{(i_{01}, i_{02}), (i_{11}, i_{12})\}$ be a parameter set of the form (5.13) associated with the basis of PCP $\{Z_0, Z_1\}$. In this case the set of nodes corresponds to

$$X_2 = \{x_1, x_2\} = \left\{\frac{i_{02}}{i_{01}}, \frac{i_{12}}{i_{11}}\right\}$$

and the corresponding Vandermonde inverse $W(x_1, x_2)$ is given by (cf. Example 5.3.1)

$$W(X_2) = \frac{1}{d}\begin{pmatrix} i_{01}i_{12} & -i_{02}i_{11} \\ -i_{01}i_{11} & i_{01}i_{11} \end{pmatrix},$$

where $d = i_{12}i_{01} - i_{02}i_{11}$. Moreover, based on (5.40) we have

$$\boldsymbol{\alpha}_{11} = -\frac{i_{12}}{d}\hat{\imath}_0 \qquad \boldsymbol{\alpha}_{12} = \frac{i_{02}}{d}\hat{\imath}_1 \qquad \boldsymbol{\alpha}_{21} = \frac{i_{11}}{d}\hat{\imath}_0 \qquad \boldsymbol{\alpha}_{22} = -\frac{i_{01}}{d}\hat{\imath}_1,$$

and therefore

$$\begin{cases} z_1 = \frac{1}{d}(-i_{12}\hat{\imath}_0 Z_0 + i_{02}\hat{\imath}_1 Z_1) \\ z_2 = \frac{1}{d}(i_{11}\hat{\imath}_0 Z_0 - i_{01}\hat{\imath}_1 Z_1) \end{cases}.$$

Example 5.4.2 Let $\Pi_2 = \{\vec{\imath}_0, \vec{\imath}_1, \vec{\imath}_2\} = \{(0,1), (i_{11}, i_{12}), (i_{21}, i_{22})\}$ be a parameter set of the form (5.18). In this case the set of nodes corresponds to

$$Y_2 = \{x_2, x_3\} = \left\{\frac{i_{12}}{i_{11}}, \frac{i_{22}}{i_{21}}\right\}$$

and the coefficients $\boldsymbol{\alpha}_{mn}$ are given by (recall (5.41))

$$\begin{array}{lll} \boldsymbol{\alpha}_{11} = -\dfrac{i_{12}i_{22}}{i_{11}i_{21}} & \boldsymbol{\alpha}_{12} = -\dfrac{1}{d}\dfrac{i_{22}}{i_{11}} & \boldsymbol{\alpha}_{13} = \dfrac{1}{d}\dfrac{i_{12}}{i_{21}} \\ \boldsymbol{\alpha}_{21} = -\dfrac{1}{2}\dfrac{c}{i_{11}i_{21}} & \boldsymbol{\alpha}_{22} = \dfrac{1}{2d}\dfrac{i_{21}}{i_{11}} & \boldsymbol{\alpha}_{23} = -\dfrac{1}{2d}\dfrac{i_{11}}{i_{21}} \\ \boldsymbol{\alpha}_{31} = -1 & \boldsymbol{\alpha}_{32} = 0 & \boldsymbol{\alpha}_{33} = 0, \end{array}$$

where $d = i_{11}i_{22} - i_{12}i_{21}$ and $c = i_{12}i_{21} + i_{11}i_{22}$. Consequently,

$$\begin{cases} z_1^2 & = -\frac{i_{12}i_{22}}{i_{11}i_{21}} Z_0^2 - \frac{1}{d}\frac{i_{22}}{i_{11}} Z_1^2 + \frac{1}{d}\frac{i_{12}}{i_{21}} Z_2^2 \\ z_1 \times z_2 & = \frac{c}{2i_{11}i_{22}} Z_0^2 + \frac{1}{2d}\frac{i_{21}}{i_{11}} Z_1^2 - \frac{1}{2d}\frac{i_{11}}{i_{21}} Z_2^2 \\ z_2^2 & = -Z_0^2 \end{cases}.$$

We recall now the case $n = 2$ of the Taylor series expansion of a holomorphic function f in a neighborhood of the origin (cf. (1.37))

$$f(\vec{z}) = \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{s=0}^{k} \frac{\partial^k f(\vec{0})}{\partial x_1^{k-s} \partial x_2^s} z_1^{k-s} \times z_2^s. \tag{5.43}$$

We observe that, in the particular case when f coincides with the PCP, the corresponding Taylor coefficients can be obtained easily from the matrix $\boldsymbol{A_k}$ in (5.12). In such case their explicit expression is given by

$$a_{s+1,t+1} = \binom{k}{t} i_{s1}^{k-t} i_{s2}^{t} \hat{\imath}_s^k, \quad t = 0, \ldots, k. \tag{5.44}$$

Our objective now is to derive a series expansion similar to (5.43), but in terms of PCP. To solve this problem we use two different approaches, both based on expression (5.39). The first idea is to replace in (5.43), $z_1^{k-s} \times z_2^s$ by the expression (5.39). Concretely we have

Proposition 5.4.3 *Any holomorphic function* $f : \Omega \subset \mathbb{R}^3 \longmapsto \mathbb{H}$ *can be written in the form*

$$f(z_1, z_2) = \sum_{k=0}^{\infty} \sum_{t=0}^{k} \beta_t^k Z_t^k, \tag{5.45}$$

where

$$\beta_t^k = \frac{1}{k!} \sum_{s=0}^{k} \binom{k}{s} \frac{\partial^k f(0)}{\partial x_1^{k-s} \partial x_2^s} \boldsymbol{\alpha}_{s+1,t+1}, \quad t = 0, \dots, k, \tag{5.46}$$

and $\boldsymbol{\alpha}_{mn}$ *are the coefficients* (5.40) *or* (5.41), *depending on the chosen parameter set* Π_k *associated with the basis* $\{Z_s^k\}_{s=0}^k$.

Concerning the convergence of the power series expansion (5.45), we can derive the following result.

Proposition 5.4.4 *If* $M_k = \max\{|\beta_t^k| : \ t = 0, \dots, k\}$ *then the radius of convergence of the series* (5.45) *is given by*

$$R = \Big(\overline{\lim_{k \to +\infty}} \sqrt[k]{M_k} \Big)^{-1}. \tag{5.47}$$

Proof. Recall the well known Cauchy-Schwartz inequality in $\mathbb{R}^2$

$$(a_1 b_1 + a_2 b_2)^2 \le (a_1^2 + a_2^2)(b_1^2 + b_2^2),$$

with (a_1, a_2) and (b_1, b_2) being vectors in $\mathbb{R}^2$. Since $|Z_s|^2 = Z_s \overline{Z_s} = x_0^2 + x_{\vec{i}_s}^2$, one obtains

$$|Z_s|^2 \le x_0^2 + (x_1^2 + x_2^2)(i_{s1}^2 + i_{s2}^2) = x_0^2 + x_1^2 + x_2^2 = |x|^2.$$

Based on this, the following estimation for each PCP holds

$$|Z_s^k| = |Z_s|^k \le |x|^k, \tag{5.48}$$

and consequently,

$$|f(z_1, z_2)| \le \sum_{k=0}^{+\infty} \sum_{t=0}^{k} |\beta_t^k| |x|^k \le \sum_{k=0}^{+\infty} (k+1) M_k |x|^k. \tag{5.49}$$

Finally, the radius of convergence R is given by

$$\frac{1}{R} = \overline{\lim_{k \to +\infty}} \sqrt[k]{M_k (k+1)} = \overline{\lim_{k \to +\infty}} \sqrt[k]{M_k},$$

which proves the assertion. ∎

A second series expansion can be obtained by replacing z_1 and z_2 in (5.43) by a linear combination of a first degree PCP basis which follows from Theorem 5.4.1.

Proposition 5.4.5 *A function $f : \Omega \subset \mathbb{R}^3 \longmapsto \mathbb{H}$ holomorphic in the neighborhood of the origin can be written in the form*

$$f(z_1, z_2) = \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{s=0}^{k} \frac{\partial^k f(\vec{0})}{\partial x_1^{k-s} \partial x_2^s} \Big(\boldsymbol{\alpha}_{11} Z_0 + \boldsymbol{\alpha}_{12} Z_1 \Big)^{k-s} \times \Big(\boldsymbol{\alpha}_{21} Z_0 + \boldsymbol{\alpha}_{22} Z_1 \Big)^{s}, \tag{5.50}$$

where $\boldsymbol{\alpha}_{mn}$ are the coefficients (5.40) *or* (5.41) *depending on the parameter set $\Pi_1 = \{\vec{\imath}_0, \vec{\imath}_1\}$ associated with the basis $\{Z_0, Z_1\}$.*

Remark 5.4.6 A similar reasoning to the one used in the proof of Proposition 5.4.4 can be applied to prove that the radius of convergence of the series

$$f(z_1, \dots, z_n) = \sum_{k=0}^{\infty} \sum_{t=1}^{N_k} \beta_t^k(n) Z^k_{\vec{\imath}_t^{(n)}},$$

is given by (5.47). In this case $N_k = \binom{n+k-1}{k}$ corresponds to the dimension of $\mathcal{M}_k^n(\mathcal{C}\ell_{0,n}, \mathcal{C}\ell_{0,n})$ (cf. [100]), which for $n = 2$ is given by $\binom{k+1}{k} = k + 1$.

The determination of conditions such that a set of N_k PCP forms a basis for the space $\mathcal{M}_k^n(\mathcal{C}\ell_{0,n}, \mathcal{C}\ell_{0,n})$ was not carried out in this work. These conditions correspond to the generalization of the non-collinear property of the $k+1$ vectors in the parameter set Π_k, in the case when $n = 2$. ♦

5.5 Numerical examples in $\mathbb{R}^3$ - implementation details

The choice of a parameter set Π_k of order k (recall (5.6)), is a determinant step for constructing basis of PCP. The $n = 2$ case of the expressions (4.25) and (4.26), i.e.

$$\vec{\imath}_s = (\cos \alpha_s, \sin \alpha_s) \tag{5.51}$$

and

$$\vec{\imath}_s = \left(\frac{1}{\sqrt{1+\alpha_s^2}}, \frac{\alpha_s}{\sqrt{1+\alpha_s^2}} \right), \tag{5.52}$$

where $\alpha_s \in \mathbb{R}$, are examples of explicit expressions for the unit vectors $\vec{\imath}_s \in \Pi_k$. We call attention to the fact that a first and natural idea for constructing Π_k corresponds to the use of the $k+1$ roots of unity

$$\omega_s^k = e^{\frac{2\pi i s}{k+1}} \cong \left(\cos \frac{2\pi s}{k+1}, \sin \frac{2\pi s}{k+1} \right), \qquad s = 0, 1, \dots, k. \tag{5.53}$$

However, it is clear that, when k is odd, the set $\{\omega_s^k\}_{s=0}^k$ is not a parameter set, since

$$\omega_s^k = -\omega^k_{s+\frac{k-1}{2}}, \quad s = 0, \dots, \tfrac{k-1}{2}.$$

Choosing parameter sets

Our purpose here is to construct several basis of PCP in $\mathbb{R}^3$, each one conveniently adapted to the problems that are of practical interest for this thesis. For this reason we present now, in detail, particular sets of vectors depending on the choice of α_s in (5.51) and (5.52), which, in turn, lead to special sets of nodes in (5.17) and (5.22). More precisely, we consider:

Choice 1: $\vec{\imath}_s = (\cos\alpha_s, \sin\alpha_s)$, where

i. $\alpha_0, \ldots, \alpha_k$ are equidistant points in $]-\frac{\pi}{2}, \frac{\pi}{2}]$

$$\alpha_s = -\frac{\pi}{2} + (s+1)\frac{\pi}{k+1}, \quad s = 0, \ldots, k. \tag{5.54}$$

ii. $\alpha_0, \ldots, \alpha_k$ is a geometric sequence with common ration $\frac{1}{2}$

$$\alpha_s = \frac{\pi}{2^s}, \quad s = 0, \ldots, k. \tag{5.55}$$

iii. $\alpha_0, \ldots, \alpha_k$ are special points in a *block structure* of the form

$$\begin{cases} \alpha_0 = & \frac{\pi}{2}, \\ \alpha_s = & \frac{\pi}{2} - \frac{(2m'+1)\pi}{2^m}, \quad s = 1, \ldots, k, \end{cases} \tag{5.56}$$

where m and m' are integers such that $s = 2^{m-1} + m'$, for $0 \leq m' \leq 2^{m-1} - 1$.

Choice 2: $\vec{\imath}_s = \left(\frac{1}{\sqrt{1+\alpha_s^2}}, \frac{\alpha_s}{\sqrt{1+\alpha_s^2}}\right)$, where

i. $\alpha_0, \ldots, \alpha_k$ are equidistant points in $[1, k+1]$

$$\alpha_s = s + 1, \quad s = 0, \ldots, k. \tag{5.57}$$

ii. $\alpha_0, \ldots, \alpha_k$ are equidistant points in $[-1, 1]$

$$\alpha_s = \frac{2s-k}{k}, \quad s = 0, \ldots, k. \tag{5.58}$$

iii. $\alpha_0, \ldots, \alpha_k$ are the Chebyshev nodes

$$\alpha_s = \cos\frac{2s+1}{2(k+1)}\pi, \quad s = 0, \ldots, k. \tag{5.59}$$

For details about these choices we refer to Tables 5.1 and 5.2, which show each associated parameter set of order 6, i.e. $\Pi_6 = \{\vec{\imath}_0, \ldots, \vec{\imath}_6\}$.

To argue that the sets $\{\vec{\imath}_s\}_{s=0}^{k}$ associated with the real numbers (5.54)-(5.59), define indeed a parameter set, we observe the following. Both expressions (5.54) and (5.56) lead to $k+1$ different arguments in $]-\frac{\pi}{2}, \frac{\pi}{2}]$, while (5.55) corresponds to $k+1$ different arguments in $[0, \pi[$. Thus, the corresponding $k+1$ unit vectors are pairwise non-collinear.

Concerning Choice 2, since the $k+1$ real numbers α_s are all distinct and

$$\alpha_r \neq \alpha_s \quad \Leftrightarrow \quad \frac{i_{s2}}{i_{s1}} \neq \frac{i_{r2}}{i_{r1}},$$

it is clear that the vectors in Π_k are pairwise non-collinear.

Choice 1

α_s	$s = 0, 1, \ldots, 6$
$i.$	$\{-\frac{5\pi}{14}, -\frac{3\pi}{14}, -\frac{\pi}{14}, \frac{\pi}{14}, \frac{3\pi}{14}, \frac{5\pi}{14}, \frac{7\pi}{14}\}$
$ii.$	$\{\pi, \frac{\pi}{2}, \frac{\pi}{4}, \frac{\pi}{8}, \frac{\pi}{16}, \frac{\pi}{32}, \frac{\pi}{64}\}$
$iii.$	$\{\frac{\pi}{2}, 0, \frac{\pi}{4}, -\frac{\pi}{4}, \frac{3\pi}{8}, \frac{\pi}{8}, -\frac{\pi}{8}\}$
$\vec{\imath}_s$	
$i.$	$\{(\cos\frac{-5\pi}{14}, \sin\frac{-5\pi}{14}), (\cos\frac{-3\pi}{14}, \sin\frac{-3\pi}{14}), (\cos\frac{-\pi}{14}, \sin\frac{-\pi}{14}), (\cos\frac{\pi}{14}, \sin\frac{\pi}{14}),$ $(\cos\frac{3\pi}{14}, \sin\frac{3\pi}{14}), (\cos\frac{5\pi}{14}, \sin\frac{5\pi}{14}), (\cos\frac{\pi}{2}, \sin\frac{\pi}{2})\}$
$ii.$	$\{(\cos\pi, \sin\pi), (\cos\frac{\pi}{2}, \sin\frac{\pi}{2}), (\cos\frac{\pi}{4}, \sin\frac{\pi}{4}), (\cos\frac{\pi}{8}, \sin\frac{\pi}{8}),$ $(\cos\frac{\pi}{16}, \sin\frac{\pi}{16}), (\cos\frac{\pi}{32}, \sin\frac{\pi}{32}), (\cos\frac{\pi}{64}, \sin\frac{\pi}{64})\}$
$iii.$	$\{(\cos\frac{\pi}{2}, \sin\frac{\pi}{2}), (\cos 0, \sin 0), (\cos\frac{\pi}{4}, \sin\frac{\pi}{4}), (\cos\frac{-\pi}{4}, \sin\frac{-\pi}{4}),$ $(\cos\frac{3\pi}{8}, \sin\frac{3\pi}{8}), (\cos\frac{\pi}{8}, \sin\frac{\pi}{8}), (\cos\frac{-\pi}{8}, \sin\frac{-\pi}{8})\}$

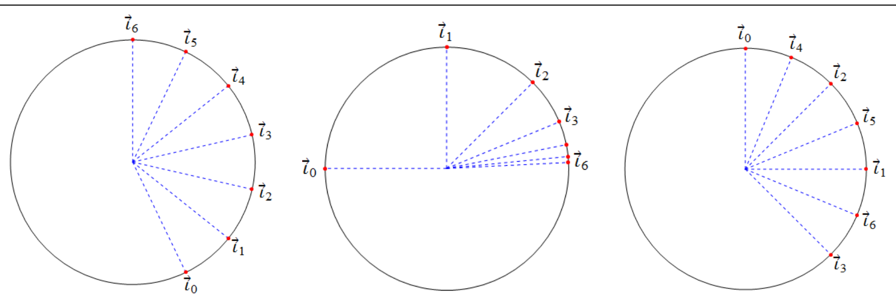

Table 5.1: Details about the parameter sets Π_6 associated with (5.54)-(5.56)

Choice 2

α_s	$s = 0, 1, \ldots, 6$
i.	$\{0, 1, 2, 3, 4, 5, 6\}$
ii.	$\left\{ -1, -\frac{2}{3}, -\frac{1}{3}, 0, \frac{1}{3}, \frac{2}{3}, 1 \right\}$
iii.	$\left\{ \frac{\pi}{14}, \frac{3\pi}{14}, \frac{5\pi}{14}, \frac{\pi}{2}, \frac{9\pi}{14}, \frac{11\pi}{14}, \frac{13\pi}{14} \right\}$
$\vec{\imath}_s$	
i.	$\left\{ (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}), (\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}), (\frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}}), (\frac{1}{\sqrt{17}}, \frac{4}{\sqrt{17}}), (\frac{1}{\sqrt{26}}, \frac{5}{\sqrt{26}}), (\frac{1}{\sqrt{37}}, \frac{6}{\sqrt{37}}), (\frac{1}{\sqrt{50}}, \frac{7}{\sqrt{50}}) \right\}$
ii.	$\left\{ (\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}), (\frac{3}{\sqrt{13}}, -\frac{2}{\sqrt{13}}), (\frac{3}{\sqrt{10}}, -\frac{1}{\sqrt{10}}), (1, 0), (\frac{3}{\sqrt{10}}, \frac{1}{\sqrt{10}}), (\frac{3}{\sqrt{13}}, \frac{2}{\sqrt{13}}), (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) \right\}$
iii.	$\left\{ (\frac{1}{\sqrt{1+\cos^2\frac{\pi}{14}}}, \frac{\cos\frac{\pi}{14}}{\sqrt{1+\cos^2\frac{\pi}{14}}}), (\frac{1}{\sqrt{1+\cos^2\frac{3\pi}{14}}}, \frac{\cos\frac{3\pi}{14}}{\sqrt{1+\cos^2\frac{3\pi}{14}}}), (\frac{1}{\sqrt{1+\cos^2\frac{5\pi}{14}}}, \frac{\cos\frac{5\pi}{14}}{\sqrt{1+\cos^2\frac{5\pi}{14}}}), (1, 0), (\frac{1}{\sqrt{1+\cos^2\frac{9\pi}{14}}}, \frac{\cos\frac{9\pi}{14}}{\sqrt{1+\cos^2\frac{9\pi}{14}}}), (\frac{1}{\sqrt{1+\cos^2\frac{11\pi}{14}}}, \frac{\cos\frac{11\pi}{14}}{\sqrt{1+\cos^2\frac{11\pi}{14}}}), (\frac{1}{\sqrt{1+\cos^2\frac{13\pi}{14}}}, \frac{\cos\frac{13\pi}{14}}{\sqrt{1+\cos^2\frac{13\pi}{14}}}) \right\}$

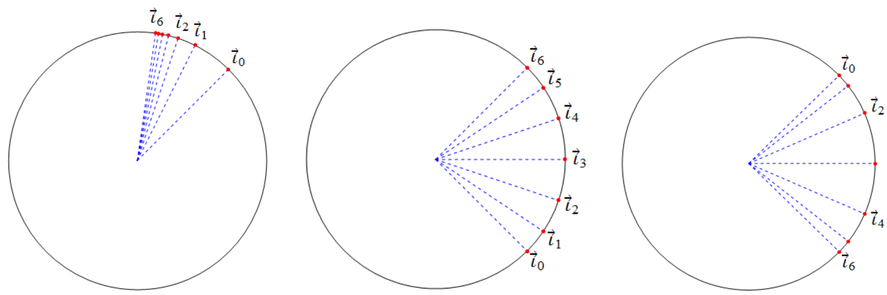

Table 5.2: Details about the parameter sets Π_6 associated with (5.57)-(5.59)

At this stage it is important to distinguish two aspects. If we want to use basis of PCP in order to benefit from their properties (see Sections 4.2 and 4.3), then we can look for parameter sets whose expressions are, in some sense, *easy to handle*. Thus, the vectors in the polar form (5.51), which can be associated with a primitive root of unity of certain order, assume an important role. Moreover, depending on the choice of arguments we make, one can obtain parameter sets with some sort of symmetry which is reflected by the parity of the trigonometric functions involved (see Choice 1 i. and 1 iii.).

On the other hand if we are interested in expressing the generalized powers in terms of a basis of PCP, then we have to invert a Vandermonde matrix associated with the nodes of the form (5.17) or (5.22) (see Sections 5.2 and 5.4). Based on this, depending on the type of vectors we choose, i.e. (5.51) or (5.52), we obtain the following types of nodes

$$x_{s+1} = \tan \alpha_s, \quad \alpha_s \neq \tfrac{\pi}{2} + k\pi, \;\; k \in \mathbb{Z} \tag{5.60}$$

or

$$x_{s+1} = \alpha_s, \quad s = 0, \ldots, k. \tag{5.61}$$

The simple form of the nodes (5.61) suggest that in this case the inversion of the referred matrix may be *computationally more efficient* than in the case of (5.60). In addition, if we are interested in inverting *all* Vandermonde matrices of order less or equal than k, then it seems natural to ask for parameter sets which fulfill the property of adding, for each k, only one new unit vector to the parameter set of order $k-1$. The particular choices (5.55), (5.56) and (5.57) also fulfill this relevant, from the computational point of view, property.

Finally we refer that the parameter set associated with the Choice 1 iii. has a *block structure* such that for $k = 2^m - 1$, the $k + 1 = 2^m$ arguments correspond to 2^m equally distributed points in $]-\frac{\pi}{2}, \frac{\pi}{2}]$. This type of parameter set is of great relevance in finding a new combinatorial identity in Section 6.2 and for this reason we decided to include also this type of nodes in this section.

PCP versus GP - implementation considerations

We focus now on the implementation details of PCP of degree k associated with the *equidistant nodes* (EN) (5.57). Our aim here is to show that, from the computational point of view, these polynomials can be a *good* alternative to GP, since they can be computed by a less time-consuming algorithm.

It is clear that the computational time and effort needed to carry out each polynomial depend on numerous parameters, such as the software used, the structure and complexity of the polynomials, the technique/algorithm implemented for their calculation, among others.

To approach the problem of computing PCP and GP bases, we make use of *Maple* and *Mathematica* systems. The option for using these two systems came from several reasons.

1. They are currently among the most powerful and popular symbolic and/or numerical mathematics softwares;

2. Both *Maple* and *Mathematica* are extensible systems;

3. The packages `Quat` for*Maple* and `Quaternions` for *Mathematica* described in Section 1.4 are free and contain functionalities for quaternionic manipulation.

For both systems we have implemented two natural solutions: the first one is based on recursion, while the second one uses iteration loops. The details about these two approaches are as follows.

Recursion

We recall that each PCP associated with the EN can be written as

$$Z_s^k = (x_0 + \hat{\imath}_s y_s)^k, \qquad s = 0, \ldots, k, \tag{5.62}$$

where

$$\hat{\imath}_s = \frac{1}{\sqrt{1+(s+1)^2}} e_1 + \frac{s+1}{\sqrt{1+(s+1)^2}} e_2 \tag{5.63}$$

and

$$y_s = \frac{1}{\sqrt{1+(s+1)^2}} x_1 + \frac{s+1}{\sqrt{1+(s+1)^2}} x_2, \tag{5.64}$$

Therefore, PCP can be obtained recursively, via

$$\begin{aligned} Z_s^0 &= 1, \\ Z_s^1 &= x_0 + \hat{\imath}_s y_s, \\ Z_s^k &= Z_s^{k-1} Z_s^1, \quad s = 0, \ldots, k. \end{aligned} \tag{5.65}$$

On the other hand, denoting by G_s^k the polynomial $z_1^{k-s} \times z_2^s$ and recalling (5.8), one can write the following recursive procedure to obtain GP:

$$\begin{aligned} G_0^0 &= 1, \\ G_s^k &= \frac{1}{k}\left((k-s) z_1 G_s^{k-1} + s\, z_2 G_{s-1}^{k-1}\right), \quad s = 0, \ldots, k. \end{aligned} \tag{5.66}$$

Programming recursively makes it easier to write simple and elegant programs to obtain both bases of GP and PCP via the referred recursive formulas. However, when the degree k increases, computing GP in terms of recursion reveals one of the potential drawbacks of recursive programs. The issue is that for the computation of GP of degree k, the value of some of the GP of lower degree will be needed many times. This means that the time required to compute GP grows exponentially with k.

In fact, both recursion solutions (5.65) and (5.66) are natural but inefficient, since many identical recursive calls are made during any given calculation. In computer science, this property is known as *overlapping subproblems*. This issue can be observed in Figures 5.1 and 5.2, where the CPU time, in seconds, is presented for both systems and for several values

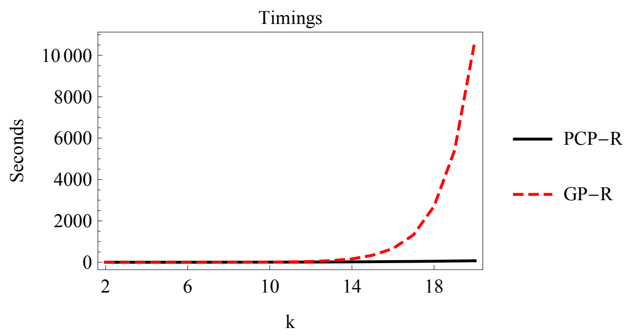

Figure 5.1: CPU time: PCP versus GP - recursion in *Mathematica*

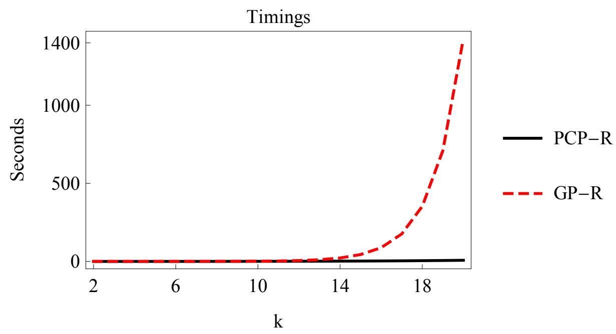

Figure 5.2: CPU time: PCP versus GP - recursion in *Maple*

of k. These values correspond to the CPU time required to construct *all* the $k+1$ linearly independent polynomials of degree k of the form (5.65) and (5.66).

Iterative loops

Problems having the overlapping subproblems property are almost always solved using *dynamic programming*, a catch-all term for any algorithm in which the definition of a function is extended as the computation proceeds. Dynamic programming is a technique for avoiding the repeated computation of the same values in a recursive program. Each value computed is immediately stored. If the value is needed again, it is not computed but simply looked up in memory (cf. [141]).

The ability to add rules to a function as the function executes makes result caching (RC) very easy to implement in *Mathematica* ([141]). Figure 5.3 contains the costs of implementing (5.66) and (5.65) in *Mathematica*, by using the aforementioned *memorization* technique.

Since the approach of *remembering values* in the recursive *Mathematica* algorithm is mostly equivalent to an iterative loop, we decided to include only the results for the first one.

Concerning the *Maple* performance of the loop implementation, Figure 5.4 illustrates clearly the advantages[4] of this procedure comparing with the recursion technique. In this way, these experiments also show evidences that it is possible to compute GP more efficiently.

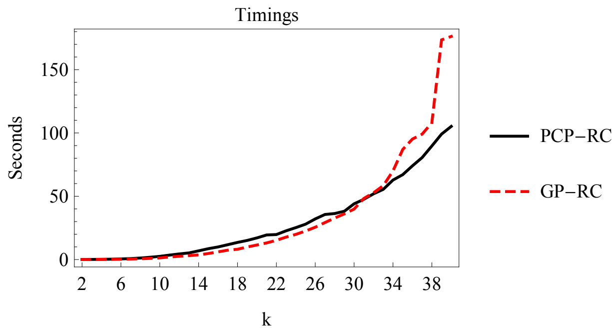

Figure 5.3: CPU time: PCP versus GP - recursion with result caching in *Mathematica*

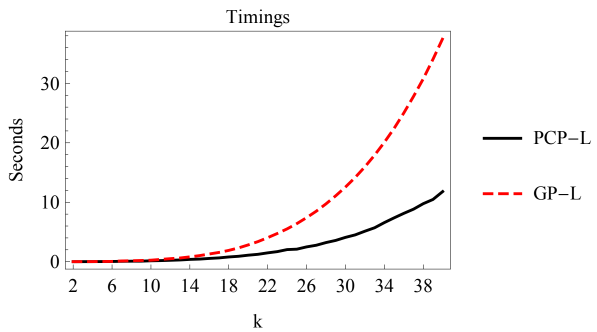

Figure 5.4: CPU time: PCP versus GP - iterative loops in *Maple*

Although in both systems and for both techniques, the PCP are less time consuming than the GP, neither of the two aforementioned alternative solutions take advantages of the simple structure of PCP.

This fact was the starting point for looking for a new algorithm to compute PCP which takes into account the isomorphism between PCP and complex powers. In other words, in what follows we propose a new algorithm which is based on complex arithmetic.

[4]The Maple software package `Quat` contains a database of GP in $\mathbb{R}^3$ and $\mathbb{R}^4$, created with the purpose of "avoiding" the time-consuming recursive algorithm implemented in the package.

Algorithm 5.5.1 [Algorithm for obtaining PCP using complex arithmetic]

1. `Choose a parameter set` Π_k `of the form` (5.6).

2. `Compute, by using complex arithmetic,` $(x_0 + yi)^k, \quad x_0, y \in \mathbb{R}$.

3. `Replace the real parameter` y `by`

$$y \leftarrow i_{s1}x_1 + i_{s2}x_2;$$

   `where` $\vec{\imath}_s = (i_{s1}, i_{s2}) \in \Pi_k$.

4. `Replace the imaginary unit` i `by the unit vector`

$$i \leftarrow i_{s1}e_1 + i_{s2}e_2; \quad s = 0, 1, \ldots, k.$$

We remark the fact that the most demanding operation is performed in Step 2 and is done in complex arithmetic. As a consequence, the time consumed for obtaining polynomials of a certain degree is substantially improved when comparing with the one needed for obtaining the same polynomials via recursion or via iterative loops. This fact is visible in Figures 5.5 and 5.6 where the performance of the algorithm for computing all the $k+1$ PCP of degree k associated with the EN is illustrated.

Moreover, for the sake of better visibility, we compare in Figures 5.7 and 5.8 the performance of the two best procedures to obtain PCP and GP in both systems. In concrete, in Figure 5.7 we compare the *Mathematica* implementation of PCP-RC (recursion with result caching), PCP-CA (Algorithm 5.5.1) and GP-RC. On the other hand, in Figure 5.8 we compare the *Maple* implementation of PCP-L (iterative loops), PCP-CA (Algorithm 5.5.1) and GP-L.

Finally, we would like to underline that the performance of *Maple* and *Mathematica* programs is closely related to two factors[5]: the structure of the associated packages `Quat` and `Quaternions` and our personal implementation of the algorithms. It was not in any way our intention to claim that either *Maple* or *Mathematica* is better than the other.

Pseudo-Complex Powers versus Generalized Powers - algorithms for inverting Vandermonde matrices

We shall now proceed with the problem of expressing each GP of degree $k-1$ in terms of the PCP associated with the equidistant nodes in $[1,k]$ given by (5.57) (EN). As has been

[5] We believe that the versions of the software we have used do not play an important role in the results.

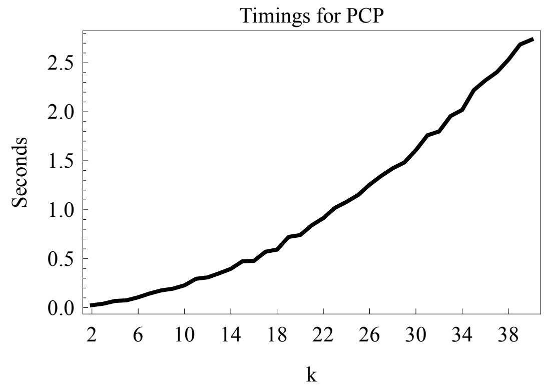

Figure 5.5: CPU time: PCP - complex arithmetic in *Mathematica*

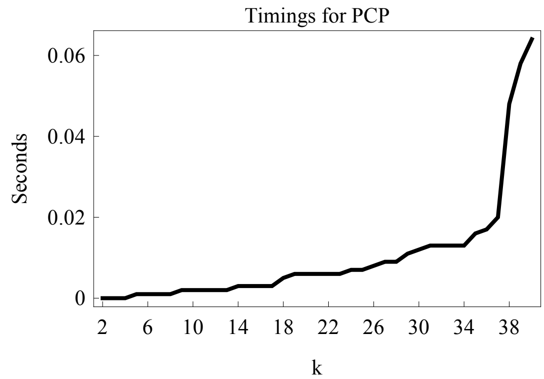

Figure 5.6: CPU time: PCP - complex arithmetic in *Maple*

explained in Section 5.4, this problem involves the inversion of the Vandermonde matrix associated with the chosen set of nodes. The reason for choosing such a set is mainly due to the fact that formula (5.28) for obtaining the inverse of the corresponding Vandermonde matrix is particularly easy to handle, since closed formulas for the symmetric polynomials $\sigma(k, s)$ and the barycentric weights $\phi(i, j)$ are available.

The implementation of this simplified formula relies on an algorithm proposed by Eisinberg and Fedele [52] which seems to be competitive with the existing ones in terms of numerical accuracy and computational effort (see e.g. [15, 117, 136] for details about other popular methods). Furthermore, the idea expressed in Section 5.3 of obtaining recursively the inverse of Vandermonde matrices is now reconsidered and numerical experiments comparing the computational effort of the Eisinberg and Fedele (EF) algorithm with a new algorithm to implement (5.34) are presented.

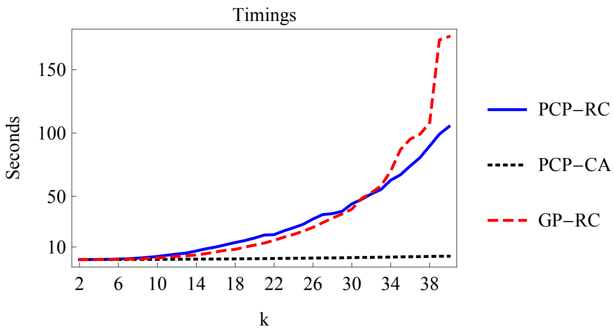

Figure 5.7: CPU time: PCP versus GP - *Mathematica* implementation

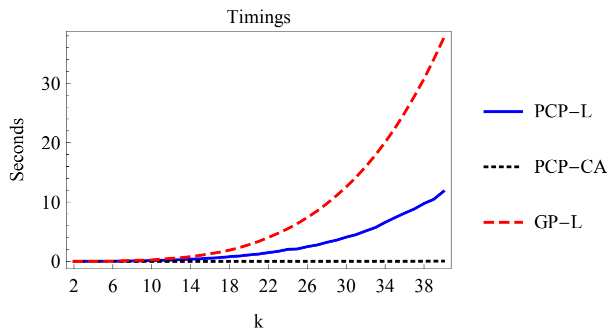

Figure 5.8: CPU time: PCP versus GP - *Maple* implementation

The EF algorithm is based on the following particular version of Theorem 5.3.5.

Proposition 5.5.1 *The inverse of the Vandermonde matrix $V(1,\ldots,k)$ has as its generic element*

$$w_{ij} = (-1)^{i+j}\Phi(k,j)\sum_{r=0}^{k-i}(-1)^r j^r \sigma(k,k-i-r), \tag{5.67}$$

where

$$\Phi(k,j) = \frac{1}{(k-j)!(j-1)!} \tag{5.68}$$

and

$$\sigma(k,s) = \begin{bmatrix} k+1 \\ k+1-s \end{bmatrix}. \tag{5.69}$$

Here $\left[{a \atop b}\right]$ denotes the usual[6] *(unsigned) Stirling numbers of the first kind [90].*

[6] $\left[{a \atop b}\right]$ gives the number of permutations of a elements that contain exactly b cycles. In [74] one can find a list of the many equivalent notations used for the Stirling numbers of first kind, while in [89] important historical remarks and identities involving these numbers are made.

Proof. The result follows from Theorem 5.3.5, using the closed expression of $\sigma(k,s)$ and $\Phi(k,j)$ presented in [52]. ■

The algorithm proposed in [52] which is in fact a modification of [117], can be rewritten in the case of EN as follows:

Algorithm 5.5.2 [MPEF algorithm for inverting $V(X_k)$ on EN]

1. Compute, by means of (5.68), $\phi(k,j),\ j=1,\ldots,k$;

2. Compute, by means of (5.69), $\sigma(k,s),\ s=0,1,2,\ldots,k$;

3. Construct the auxiliary function

$$\begin{aligned}
&\psi(1,j) = (-1)^{1+j}\frac{\sigma(k,k)}{j},\ j=1,\ldots,k\\
&\psi(2,j) = -\psi(1,j)\sum_{\substack{r=1\\ r\neq j}}^{k}\frac{1}{r},\ j=1,\ldots,k\\
&\psi(k,j) = (-1)^{k+j},\ j=1,\ldots,k\\
&\psi(i-1,j) = j\psi(i,j) - (-1)^{i+j}\sigma(k,k+1-i),\ i=k,k-1,\ldots,2, \quad j=1,2,\ldots,k;
\end{aligned}$$

4. Compute the $j-$th column $[\psi(i,j)\phi(k,j)],\ j=1,2,\ldots,k.$

When X_k is the set of EN we observe that

$$X_k = \{x_1,x_2,\ldots,x_{k-1}\}\bigcup\{x_k\} = X_{k-1}\bigcup\{x_k\}. \tag{5.70}$$

This fact, together with the closed formulas (5.68) and (5.69), suggest that the computation of the k matrices

$$\left\{V^{-1}(X_1),V^{-1}(X_2),\ldots,V^{-1}(X_k)\right\}$$

can be possibly made in a more efficient way than when computing the referred matrices by using Algorithm 5.5.2. Based on this observation and on Lemma 5.3.6 we propose the following recursive formula for the generic entry w_{ij}^k of $W(X_k)=V^{-1}(X_k)$.

Proposition 5.5.2 *Let $X_k=\{1,\ldots,k\}$, then the generic entry of $W(X_k)$ has the following form*

1. For $j = 1, \ldots, k$

$$w_{1j}^k = (-1)^{j+1} \binom{k}{j} \tag{5.71}$$

$$w_{kj}^k = \frac{(-1)^{k+j}}{(k-j)!(j-1)!} \tag{5.72}$$

2. For $i = 2, \ldots, k-1$

$$w_{ik}^k = w_{i+1,k}^k + \frac{(-1)^{k-i}}{(k-1)!} \begin{bmatrix} k+1 \\ i+1 \end{bmatrix} \tag{5.73}$$

$$w_{ij}^k = \frac{k}{k-j} w_{ij}^{k-1} - \frac{1}{k-j} w_{i-1,j}^{k-1}, \quad j = 1, \ldots, k-1. \tag{5.74}$$

Proof. Formula (5.74) follows immediately from (5.33).

If we replace expressions (5.68) and (5.69) into (5.28), then one obtains

$$w_{ij}^k = \frac{(-1)^{k+j}}{(k-j)!(j-1)!} \sum_{r=0}^{k-i} (-1)^r \begin{bmatrix} k+1 \\ k+1-r \end{bmatrix} j^{k-i-r}, \quad i, j = 1, \ldots, k \tag{5.75}$$

and therefore (5.72) follows straightforward. On the other hand, this last expression yields, for $i = 1$,

$$\begin{aligned} w_{1j}^k &= \frac{(-1)^{k+j}}{(k-j)!(j-1)!} \sum_{r=0}^{k-1} (-1)^r \begin{bmatrix} k+1 \\ k+1-r \end{bmatrix} j^{k-1-r} \\ &= \frac{(-1)^{k+j}}{(k-j)!(j-1)!j^2} \sum_{r=0}^{k-1} (-1)^{k+1-r} \begin{bmatrix} k+1 \\ r+2 \end{bmatrix} j^{r+2}. \end{aligned}$$

Thus, by recalling the well known properties of the Stirling numbers of the first kind

$$\begin{bmatrix} k \\ 0 \end{bmatrix} = 0, \quad \begin{bmatrix} k \\ 1 \end{bmatrix} = (k-1)!, \; k \geq 1,$$

and

$$\sum_{r=0}^{k} (-1)^{k-r} \begin{bmatrix} k \\ r \end{bmatrix} j^r = 0, \quad j = 1, \ldots, k,$$

one obtains

$$\begin{aligned} w_{1j}^k &= \frac{(-1)^{k+j}}{(k-j)!(j-1)!j^2} \left[\sum_{r=0}^{k+1} (-1)^{k+1-r} \begin{bmatrix} k+1 \\ r+2 \end{bmatrix} j^{r+2} + (-1)^{k+1} \begin{bmatrix} k+1 \\ 1 \end{bmatrix} j \right] \\ &= (-1)^{j+1} \binom{k}{j}. \end{aligned}$$

Finally, for the case where $j = k$ we note that $w^k_{kk} = \frac{1}{(k-1)!}$ and, for $i = 2, \ldots, k-1$,

$$
\begin{aligned}
w^k_{ik} &= \frac{1}{(k-1)!}\sum_{r=0}^{k-i}(-1)^r \begin{bmatrix} k+1 \\ k+1-r \end{bmatrix} k^{k-i-r} \\
&= \frac{1}{(k-1)!}\sum_{r=0}^{k-1-i}(-1)^r \begin{bmatrix} k+1 \\ k+1-r \end{bmatrix} k^{k-i-r} + \frac{(-1)^{k-i}}{(k-1)!}\begin{bmatrix} k+1 \\ k+1-(k-i) \end{bmatrix}.
\end{aligned}
$$

Therefore,

$$
w^k_{ik} = k w^k_{i+1,k} + \frac{(-1)^{k-i}}{(k-1)!}\begin{bmatrix} k+1 \\ i+1 \end{bmatrix}.
$$

■

Based on this result, we propose the following algorithm

Algorithm 5.5.3 [Modified algorithm for inverting $V(X_k)$ on EN]

1. `Compute the first and last rows of` $W(X_k)$ `by means of` (5.71) `and` (5.72), `respectively;`

2. `Compute recursively:`

$$
w^k_{ik} = w^k_{i+1,k} + \frac{(-1)^{k-i}}{(k-1)!}\begin{bmatrix} k+1 \\ i+1 \end{bmatrix};
$$

$$
w^k_{ij} = \frac{k}{k-j} w^{k-1}_{ij} - \frac{1}{k-j} w^{k-1}_{i-1,j}, \quad j = 1, \ldots, k-1,
$$

`for` $i = k-1, \ldots, 2$.

At this stage our goal is to provide an experimental comparison between Eisinberg and Fedele algorithm and the modified algorithm proposed in this work for inverting the Vandermonde matrix $V(1, \ldots, k)$.

All the numerical results were obtained by the use of *Mathematica*, working with infinite precision, and consequently we do not have to take into account roundoff errors associated with the calculation process. Figures 5.9 and 5.10 compare the performance of this new algorithm with the one proposed by Eisinberg and Fedele. We underline that the CPU time presented in Figure 5.9 concerns the effort needed to compute the inverse $W(X_k)$, for a given k. When the goal is to compute *all* the inverses of order less or equal than k we observe in Figure 5.10 that the modified algorithm is, in fact, very competitive comparing with the Eisinberg and Fedele algorithm.

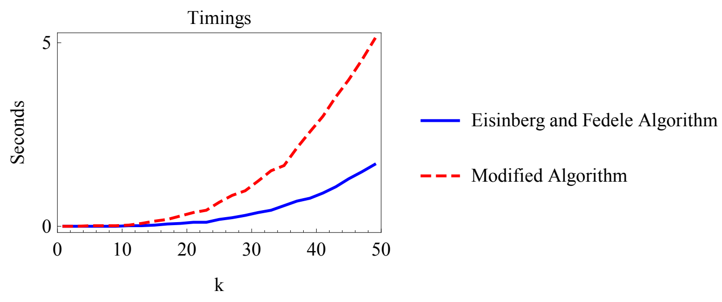

Figure 5.9: CPU time for inverting $V(X_k)$, $k = 1, \ldots, 50$

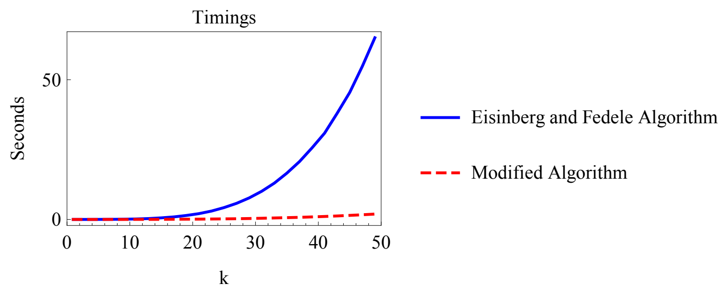

Figure 5.10: CPU time for inverting $\{V(X_1), V(X_2), \ldots, V(X_k)\}$, $k = 1, \ldots, 50$

Taylor series

We start by recalling the particular form (5.62)-(5.64) of the PCP associated with the EN and give a list of the first PCP bases written in terms of cartesian coordinates (see Table 5.3). The Taylor coefficients of the PCP associated with EN can be obtained from (5.44). Their explicit expression is given by

$$a_{s+1,t+1} = \begin{cases} (-1)^{\frac{k}{2}} \binom{k}{t} (s+1)^t \left(1+(s+1)^2\right)^{-\frac{k}{2}} & \text{if } k \text{ is even} \\ (-1)^{\frac{k-1}{1}} \binom{k}{t} (s+1)^t \left(1+(s+1)^2\right)^{-\frac{k+1}{2}} \left(e_1 + (s+1)e_2\right) & \text{if } k \text{ is odd} \end{cases} \tag{5.76}$$

Example 5.5.1 The first Taylor coefficients $a_{s+1,t+1}$ of the PCP of degree k, $(k \leq 4)$ associated with the set of EN $\{1, \ldots, k+1\}$ can be obtained from the entries of the following matrices:

$$\boldsymbol{A_1} = \begin{pmatrix} \frac{1}{2}e_1 + \frac{1}{2}e_2 & \frac{1}{2}e_1 + \frac{1}{2}e_2 \\ \frac{1}{5}e_1 + \frac{2}{5}e_2 & \frac{2}{5}e_1 + \frac{4}{5}e_2 \end{pmatrix}, \quad \boldsymbol{A_2} = \begin{pmatrix} -\frac{1}{2} & -1 & -\frac{1}{2} \\ -\frac{1}{5} & -\frac{4}{5} & -\frac{4}{5} \\ -\frac{1}{10} & -\frac{3}{5} & -\frac{9}{10} \end{pmatrix},$$

k	$Z_s^k, \quad s = 0, 1, \ldots, k$
1	$Z_0^1 = x_0 + \frac{1}{1}(x_1 + x_2)\boldsymbol{e_1} + \frac{1}{2}(x_1 + 2x_2)\boldsymbol{e_2}$ $Z_1^1 = x_0 + \frac{1}{5}(x_1 + 2x_2)\boldsymbol{e_1} + \frac{2}{5}(x_1 + 2x_2)\boldsymbol{e_2}$
2	$Z_0^2 = x_0^2 - \frac{1}{2}(x_1 + x_2)^2 + x_0(x_1 + x_2)\boldsymbol{e_1} + x_0(x_1 + 2x_2)\boldsymbol{e_2}$ $Z_1^2 = \frac{1}{5}\big((5x_0^2 - (x_1 + 2x_2)^2\big) + \frac{2}{5}x_0(x_1 + 2x_2)\boldsymbol{e_1} + \frac{4}{5}x_0(x_1 + 2x_2)\boldsymbol{e_2}$ $Z_2^2 = \frac{1}{10}\big(10x_0^2 - (x_1 + 3x_2)^2\big) + \frac{1}{5}x_0(x_1 + 3x_2)\boldsymbol{e_1} + \frac{3}{5}x_0(x_1 + 3x_2)\boldsymbol{e_2}$
3	$Z_0^3 = \frac{1}{2}x_0(2x_0^2 - 3(x_1 + x_2)^2) -$ $\quad -\frac{1}{4}(x_1 + x_2)(-6x_0^2 + (x_1 + x_2)^2)\boldsymbol{e_1} - \frac{1}{4}(x_1 + x_2)(-6x_0^2 + (x_1 + x_2)^2)\boldsymbol{e_2}$ $Z_1^3 = \frac{1}{5}x_0(5x_0^2 - 3(x_1 + 2x_2)^2) -$ $\quad -\frac{1}{25}(x_1 + 2x_2)(-15x_0^2 + (x_1 + 2x_2)^2)\boldsymbol{e_1} - \frac{2}{25}(x_1 + 2x_2)(-15x_0^2 + (x_1 + 2x_2)^2)\boldsymbol{e_2}$ $Z_2^3 = \frac{1}{10}x_0(10x_0^2 - 3(x_1 + 3x_2)^2) -$ $\quad -\frac{1}{100}(x_1 + 3x_2)(-30x_0^2 + (x_1 + 3x_2)^2)\boldsymbol{e_1} - \frac{3}{100}(x_1 + 3x_2)(-30x_0^2 + (x_1 + 3x_2)^2)\boldsymbol{e_2}$ $Z_3^3 = \frac{1}{17}x_0(17x_0^2 - 3(x_1 + 4x_2)^2) -$ $\quad -\frac{1}{289}(x_1 + 4x_2)(-51x_0^2 + (x_1 + 4x_2)^2)\boldsymbol{e_1} - \frac{4}{289}(x_1 + 4x_2)(-51x_0^2 + (x_1 + 4x_2)^2)\boldsymbol{e_2}$
4	$Z_0^4 = x_0^4 - 3x_0^2(x_1 + x_2)^2 + \frac{1}{4}(x_1 + x_2)^4 -$ $\quad -x_0(x_1 + x_2)(-2x_0^2 + (x_1 + x_2)^2)\boldsymbol{e_1} - x_0(x_1 + x_2)(-2x_0^2 + (x_1 + x_2)^2)\boldsymbol{e_2}$ $Z_1^4 = \frac{1}{25}\big(25x_0^4 - 30x_0^2(x_1 + 2x_2)^2 + (x_1 + 2x_2)^4\big) -$ $\quad -\frac{4}{25}x_0(x_1 + 2x_2)(-5x_0^2 + (x_1 + 2x_2)^2)\boldsymbol{e_1} - \frac{8}{25}x_0(x_1 + 2x_2)(-5x_0^2 + (x_1 + 2x_2)^2)\boldsymbol{e_2}$ $Z_2^4 = \frac{1}{100}\big(100x_0^4 - 60x_0^2(x_1 + 3x_2)^2 + (x_1 + 3x_2)^4\big) -$ $\quad -\frac{1}{25}x_0(x_1 + 3x_2)(-10x_0^2 + (x_1 + 3x_2)^2)\boldsymbol{e_1} - \frac{3}{25}x_0(x_1 + 3x_2)(-10x_0^2 + (x_1 + 3x_2)^2)\boldsymbol{e_2}$ $Z_3^4 = \frac{1}{289}\big(289x_0^4 - 102x_0^2(x_1 + 4x_2)^2 + (x_1 + 4x_2)^4\big) -$ $\quad -\frac{4}{289}x_0(x_1 + 4x_2)(-17x_0^2 + (x_1 + 4x_2)^2)\boldsymbol{e_1} - \frac{16}{289}x_0(x_1 + 4x_2)(-17x_0^2 + (x_1 + 4x_2)^2)\boldsymbol{e_2}$ $Z_4^4 = \frac{1}{676}\big(676x_0^4 - 156x_0^2(x_1 + 5x_2)^2 + (x_1 + 5x_2)^4\big) -$ $\quad -\frac{1}{169}x_0(x_1 + 5x_2)(-26x_0^2 + (x_1 + 5x_2)^2)\boldsymbol{e_1} - \frac{5}{169}x_0(x_1 + 5x_2)(-26x_0^2 + (x_1 + 5x_2)^2)\boldsymbol{e_2}$

Table 5.3: The first PCP on EN expressed in $\{x_0, x_1, x_2\}$

$$\boldsymbol{A_3} = \begin{pmatrix} -\frac{1}{4}e_1 - \frac{1}{4}e_2 & -\frac{3}{4}e_1 - \frac{3}{4}e_2 & -\frac{3}{4}e_1 - \frac{3}{4}e_2 & -\frac{1}{4}e_1 - \frac{1}{4}e_2 \\ -\frac{1}{25}e_1 - \frac{2}{25}e_2 & -\frac{6}{25}e_1 - \frac{12}{25}e_2 & -\frac{12}{25}e_1 - \frac{24}{25}e_2 & -\frac{8}{25}e_1 - \frac{16}{25}e_2 \\ -\frac{1}{100}e_1 - \frac{3}{100}e_2 & -\frac{9}{100}e_1 - \frac{27}{100}e_2 & -\frac{27}{100}e_1 - \frac{81}{100}e_2 & -\frac{27}{100}e_1 - \frac{81}{100}e_2 \\ -\frac{1}{289}e_1 - \frac{4}{289}e_2 & -\frac{12}{289}e_1 - \frac{48}{289}e_2 & -\frac{48}{289}e_1 - \frac{192}{289}e_2 & -\frac{64}{289}e_1 - \frac{256}{289}e_2 \end{pmatrix},$$

and

$$\boldsymbol{A_4} = \begin{pmatrix} \frac{1}{4} & 1 & \frac{3}{2} & 1 & \frac{1}{4} \\ \frac{1}{25} & \frac{8}{25} & \frac{24}{25} & \frac{32}{25} & \frac{16}{25} \\ \frac{1}{100} & \frac{3}{25} & \frac{27}{50} & \frac{27}{25} & \frac{81}{100} \\ \frac{1}{289} & \frac{16}{289} & \frac{96}{289} & \frac{256}{289} & \frac{256}{289} \\ \frac{1}{676} & \frac{5}{169} & \frac{75}{338} & \frac{125}{169} & \frac{625}{676} \end{pmatrix}.$$

The first four PCP bases associated with the EN can now be written in terms of the GP basis, in the way illustrated in Table 5.4.

k	$Z_s^k,\ \ s=0,1,\ldots,k$
1	$Z_0^1 = \frac{1}{2}\left[(e_1+e_2)\boldsymbol{z_1} + (e_1+e_2)\boldsymbol{z_2}\right]$
	$Z_1^1 = \frac{1}{5}\left[(e_1+2e_2)\boldsymbol{z_1} + (2e_1+4e_2)\boldsymbol{z_2}\right]$
2	$Z_0^2 = -\frac{1}{2}\left[\boldsymbol{z_1^2} + 2\boldsymbol{z_1}\times\boldsymbol{z_2} + \boldsymbol{z_2^2}\right]$
	$Z_1^2 = -\frac{1}{5}\left[\boldsymbol{z_1^2} + 4\boldsymbol{z_1}\times\boldsymbol{z_2} + 4\boldsymbol{z_2^2}\right]$
	$Z_2^2 = -\frac{1}{10}\left[\boldsymbol{z_1^2} + 6\boldsymbol{z_1}\times\boldsymbol{z_2} + 9\boldsymbol{z_2^2}\right]$
3	$Z_0^3 = -\frac{1}{4}\left[(e_1+e_2)\boldsymbol{z_1^3} + (3e_1+3e_2)\boldsymbol{z_1^2}\times\boldsymbol{z_2} + (3e_1+3e_2)\boldsymbol{z_1}\times\boldsymbol{z_2^2} + (e_1+e_2)\boldsymbol{z_2^3}\right]$
	$Z_1^3 = -\frac{1}{25}\left[(e_1+2e_2)\boldsymbol{z_1^3} + (6e_1+12e_2)\boldsymbol{z_1^2}\times\boldsymbol{z_2} + (12e_1+24e_2)\boldsymbol{z_1}\times\boldsymbol{z_2^2} + (8e_1+16e_2)\boldsymbol{z_2^3}\right]$
	$Z_2^3 = -\frac{1}{100}\left[(e_1+3e_2)\boldsymbol{z_1^3} + (9e_1+27e_2)\boldsymbol{z_1^2}\times\boldsymbol{z_2} + (27e_1+81e_2)\boldsymbol{z_1}\times\boldsymbol{z_2^2} + (27e_1+81e_2)\boldsymbol{z_2^3}\right]$
	$Z_3^3 = -\frac{1}{289}\left[(e_1+4e_2)\boldsymbol{z_1^3} + (12e_1+48e_2)\boldsymbol{z_1^2}\times\boldsymbol{z_2} + (48e_1+192e_2)\boldsymbol{z_1}\times\boldsymbol{z_2^2} + (64e_1+256e_2)\boldsymbol{z_2^3}\right]$
4	$Z_0^4 = \frac{1}{4}\left[\boldsymbol{z_1^4} + 4\boldsymbol{z_1^3}\times\boldsymbol{z_2} + 6\boldsymbol{z_1^2}\times\boldsymbol{z_2^2} + 4\boldsymbol{z_1}\times\boldsymbol{z_2^3} + \boldsymbol{z_2^4}\right]$
	$Z_1^4 = \frac{1}{25}\left[\boldsymbol{z_1^4} + 8\boldsymbol{z_1^3}\times\boldsymbol{z_2} + 24\boldsymbol{z_1^2}\times\boldsymbol{z_2^2} + 32\boldsymbol{z_1}\times\boldsymbol{z_2^3} + 16\boldsymbol{z_2^4}\right]$
	$Z_2^4 = \frac{1}{100}\left[\boldsymbol{z_1^4} + 12\boldsymbol{z_1^3}\times\boldsymbol{z_2} + 544\boldsymbol{z_1^2}\times\boldsymbol{z_2^2} + 108\boldsymbol{z_1}\times\boldsymbol{z_2^3} + 81\boldsymbol{z_2^4}\right]$
	$Z_3^4 = \frac{1}{289}\left[\boldsymbol{z_1^4} + 16\boldsymbol{z_1^3}\times\boldsymbol{z_2} + 96\boldsymbol{z_1^2}\times\boldsymbol{z_2^2} + 256\boldsymbol{z_1}\times\boldsymbol{z_2^3} + 256\boldsymbol{z_2^4}\right]$
	$Z_4^4 = \frac{1}{676}\left[\boldsymbol{z_1^4} + 20\boldsymbol{z_1^3}\times\boldsymbol{z_2} + 150\boldsymbol{z_1^2}\times\boldsymbol{z_2^2} + 500\boldsymbol{z_1}\times\boldsymbol{z_2^3} + 625\boldsymbol{z_2^4}\right]$

Table 5.4: The first PCP on EN expressed in terms of the generalized powers $\{z_1^{k-s}\times z_2^s\}_{s=0}^k$

We have now the necessary tools to express a reduced quaternion valued holomorphic function in a series expansion of PCP associated with EN. In fact, Proposition 5.4.3 together with the expressions (5.40) and (5.63) produce the following result.

Proposition 5.5.3 *Any holomorphic function $f : \Omega \subset \mathbb{R}^3 \to \mathbb{H}$ can be written as a series of the form*

$$f(z_1, z_2) = \sum_{k=0}^{\infty} \sum_{t=0}^{k} \beta_t^k Z_t^k,$$

where the coefficients β_t^k are given by

$$\beta_t^k = \frac{(-e_1 - (t+1)e_2)^k}{k!} \sum_{s=0}^{k} \frac{\partial^k f(0)}{\partial x_1^{k-s} \partial x_2^s} w_{s+1,t+1}^{k+1}. \tag{5.77}$$

Remark 5.5.4 We underline that $(-e_1 - (t+1)e_2)^k$, depending on the parity of k, is a real number (if k even) or a pure vector (if k odd). More precisely,

$$(-e_1 - (t+1)e_2)^k = (-1)^{\frac{k}{2}} (t^2 + 2t + 2)^{\lfloor \frac{k}{2} \rfloor} \gamma_{k,t},$$

where $\gamma_{k,t}$ is

$$\gamma_{k,t} = \begin{cases} 1, & k \text{ even} \\ e_1 + (t+1)e_2, & k \text{ odd} \end{cases}.$$

♦

Comments to Chapter 5

Algorithms for obtaining PCP, based on the use of complex arithmetic, are not restricted to the particular case of equally spaced nodes (cf. Algorithm 5.5.1). In fact, we have performed several other experiments with the other sets of nodes mentioned in this chapter and all of them show the high (and expectable) effectiveness of the algorithm.

On the other hand, modified algorithms similar to Algorithm 5.5.3 can be used, whenever the nodes satisfy property (5.70). This is not the case, for example, of the Chebyshev nodes (5.59). However, their expression allows to obtain interesting relations which simplify the structure of $V^{-1}(X_k)$. As a direct consequence of the symmetry of these nodes, it is possible to detect a corresponding alternating symmetry on the columns of the associated Vandermonde inverse. Although several promising results were obtained for such choice, further investigation should be carried out to test the effectiveness of the algorithm. In fact, several algorithms for inverting Vandermonde matrices on Chebyshev nodes are available in the literature and therefore comparison tests should be done (cf. [53] and [71]).

The recursive algorithm here proposed for inverting Vandermonde matrices has its own value, but for general purpose applications, an error analysis has to be done, if one wants to use floating point arithmetic systems. This is an interesting topic, but it is clearly out of the scope of this dissertation.

The idea of choosing parameter sets that makes it possible to obtain PCP with prescribed properties would be of very practical interest. As expected this is not an easy or always possible task.

Chapter 6

Hypercomplex function theory and combinatorial identities

6.1 Introduction

The aim of this chapter is to show one more possibility of advantageously using the pseudo-complex powers PCP. As in the previous chapter, it throws again light upon the freedom of choosing particular parameter sets for the (PCP). Completing particular cases of combinatorial identities studied in Chapter 3, this time the PCP are the key to combinatorial identities, confirming once more their role as an interesting subject on the crossroad of HFT and Combinatorics.

In [131] A. Sofo studied a general binomial sum of the form

$$f_m(a,b) = \sum_{k=0}^{m} \binom{m}{k} (-a)^k \binom{bk}{k}$$

for real values a and integer values b. If $a = \frac{1}{2}$ and $b = 2$ then follows, depending from an even or odd value of m, that

$$f_{2m}\Big(\frac{1}{2}, 2\Big) = \sum_{k=0}^{2m} \binom{2m}{k} \Big(-\frac{1}{2}\Big)^k \binom{2k}{k} = \frac{1}{2^{2m}} \binom{2m}{m} = c_{2m}(2). \tag{6.1}$$

and

$$f_{2m+1}\Big(\frac{1}{2}, 2\Big) = \sum_{k=0}^{2m+1} \binom{2m+1}{k} \Big(-\frac{1}{2}\Big)^k \binom{2k}{k} = 0. \tag{6.2}$$

These are the famous *Reed-Dawson identities.* For the first time, they were proved in 1968 in Riordan's book [125] by a recurrence relation, while in the paper [121] the identity (6.1) was mentioned as *Knuth's Old Sum* and proved by using an Euler transformation. Sofo gave in [131] one more proof of the Reed-Dawson Identities relying also on hypergeometric functions. Until very recently they resisted direct proofs by purely combinatorial methods (cf. [2]).

Both identities (6.1) and (6.2) are examples of the huge number of combinatorial identities which involve the central binomial coefficients. But their frequent appearance in the form of the *weighted central binomial coefficients* $c_k = c_k(2)$ given in formula (2.18) seems to be more remarkable.

A further example for this fact is an identity that we call *Riordan-Sofo Identity* also proved in [131]. It relies on the close relation to *Reed-Dawson* as a consequence from the observation that $f_{2m}(\frac{1}{2}, 2) = f_m(\frac{1}{4}, 2)$. More concretely, the Riordan-Sofo Identity is in different equivalent expressions given by

$$\sum_{u=0}^{m} \frac{(-1)^u}{2^{2u}} \binom{2u}{u} \binom{m}{u} = \sum_{u=0}^{m} (-1)^u \binom{m}{u} c_{2u} = \sum_{u=0}^{m} (-1)^{m-u} \binom{m}{u} c_{2m-2u} = c_{2m}. \tag{6.3}$$

The previous chapters have shown that the coefficients $c_k(2)$ are structural constants of generalized Appell polynomials of particular importance. They are not only the values of $c_k(n)$ in Proposition 2.2.2 for $n = 2$ (cf. (2.6)), but appear also in Appell polynomials of more than three real variables as had been shown in Theorem 4.4.6.

Our observation of some connection between particular types of combinatorial identities and holomorphic Appell polynomials suggested a further study of intrinsic properties of those polynomials. As result we prove in this chapter a *Riordan-Sofo type identity* of the form

$$\sum_{u=0}^{s} (-1)^{s-u} \binom{s}{u} c_{2(m-u)} = \binom{m}{s} \binom{2m}{2s}^{-1} c_{2m} \tag{6.4}$$

where m is a positive integer and $s = 0, 1, \ldots, m$. In other words, the *Riordan-Sofo type identity* includes any value of the upper limit s of the sum.

Obviously, in the particular case of $s = m$ formula (6.4) is identical with (6.3), thereby showing that methods of HFT can successfully be used as an analytic tool for obtaining new combinatorial identities.

The *Riordan-Sofo type identity* 6.4 was for the first time presented in [38], but as an auxiliary tool for proving different representations of generalized Appell polynomials. More details can be found in Remark 6.2.3.

Our approach here relies basically on the representation of one and the same holomorphic Appell polynomial by two different sets of variables and the comparison of the corresponding coefficients. This can be considered as the application of a particular case of the bijective methods widely used in *Enumerative and Algebraic Combinatorics* (cf. [145]).

In concrete, we use the expression of the SAP (2.11) for ($n = 2$)

$$\mathcal{P}_k(x_0, \underline{x}) = \sum_{s=0}^{k} c_s(2) \binom{k}{s} x_0^{k-s} \underline{x}^s \tag{6.5}$$

in terms of generalized powers (GP) (cf. Definition 1.3.4) as well as in terms of a suitably chosen basis of pseudo-complex powers (PCP). Due to the fact that the parameter set of the PCP can be written in polar coordinates, we obtain in auxiliary steps several formulae for sums of even powers of *cosine* in closed form which, to the best of our knowledge, have not be considered so far.

Finally we list a number of formulae and facts about the *weighted central binomial coefficients* $c_k(2)$ to gather some old and new representations and properties of these special coefficients.

6.2 Hypercomplex polynomials and combinatorial identities

Our aim is to stress the potentialities of HFT in Combinatorics in the particular case of applications of holomorphic hypercomplex polynomials. Therefore we consider for the case $n = 2$, as mentioned in the previous introductory remarks, two different expressions of the polynomial (2.11), i.e. $\mathcal{P}_k(x_0, \underline{x})$ with values in the set of paravectors $\mathcal{A}_2 \subset \mathcal{C}\ell_{0,n}$.

The $\mathcal{P}_k(x_0, \underline{x}), k = 0, 1, \ldots,$ are already known from Proposition 2.2.2 in terms of the hypercomplex variables z_1 e z_2 as

$$\mathbf{P}_k(z_1, z_2) = c_k(2) \sum_{s=0}^{k} z_1^{k-s} \times z_2^s \binom{k}{s} e_1^{k-s} \times e_2^s \tag{6.6}$$

where (cf. 2.18)

$$c_k(2) = \frac{1}{2^k} \binom{k}{\lfloor \frac{k}{2} \rfloor}.$$

Examples of the first polynomials can be found in Table 2.4.

Consider now the particular basis

$$Z_s^k = Z_{\vec{i}_s}^k = (z_1 i_{s1} + z_2 i_{s2})^k (i_{s1} e_1 + i_{s2} e_2)^k, \quad s = 0, 1, \ldots, k, \tag{6.7}$$

of PCP (cf. (5.4)) associated with the parameter set Π_k labeled by Choice 1, case iii, in Section 5.5 (cf. (5.56)).

Before deriving the corresponding expression of $\mathcal{P}_k(x_0, \underline{x})$ in terms of (6.7) we explain some important details of this Π_k.

Let k' be of the form $k' = 2^n - 1$, for $n = 1, 2, \ldots$. For those values of k', the set $A^{k'}$ is the set of $k'+1$ arguments corresponding to equi-distributed points on the right half circumference (in clock wise order) given by

$$\frac{\pi}{2}, \frac{\pi}{2} - \frac{\pi}{k'+1}, \frac{\pi}{2} - \frac{2\pi}{k'+1}, \ldots, \frac{\pi}{2} - \frac{k'\pi}{k'+1}.$$

The set $A^{k''}$, with $k'' = 2^{n+1} - 1 = 2k' + 1$, is successively obtained from the set $A^{k'}$ by adding the intermediate arguments of the intervals

$$\left[\tfrac{\pi}{2} - \tfrac{\pi}{k'+1}, \tfrac{\pi}{2}\right], \; \left[\tfrac{\pi}{2} - \tfrac{2\pi}{k'+1}, \tfrac{\pi}{2} - \tfrac{\pi}{k'+1}\right], \; \ldots, \left[\tfrac{\pi}{2} - \tfrac{k'\pi}{k'+1}, \tfrac{\pi}{2} - \tfrac{(k'-1)\pi}{k'+1}\right], \; \left[-\tfrac{\pi}{2}, \tfrac{\pi}{2} - \tfrac{k'\pi}{k'+1}\right].$$

But these new points appear in a different order, if we consider the intermediate argument sets A^k between $A^{k'}$ and $A^{k'+2^n} = A^{k''}$, i.e.

$$A^{k'+1}, A^{k'+2}, \ldots, A^{k''}.$$

With $s = k' + 1 + n'$, where $0 \leq n' \leq 2^n - 1$, we obtain a sequence of 2^n indexes s by running n' from 0 until $2^n - 1$. This procedure defines the sets

$$\boldsymbol{A}^k = \{\alpha_0, \alpha_1, \ldots, \alpha_s, \ldots, \alpha_k\}, \tag{6.8}$$

where

$$\begin{cases} \alpha_0 = & \frac{\pi}{2}, \\ \alpha_s = & \frac{\pi}{2} - \frac{(2n'+1)\pi}{2^{n+1}}, \quad s = 1, \ldots, k \end{cases}, \tag{6.9}$$

and consequently, the parameter set

$$\Pi_k = \{\vec{\imath}_0, \vec{\imath}_1, \ldots, \vec{\imath}_k\} = \{(\cos\alpha_0, \sin\alpha_0), (\cos\alpha_1, \sin\alpha_1), \ldots, (\cos\alpha_k, \sin\alpha_k)\}. \tag{6.10}$$

We underline the fact that this particular choice of parameters fulfills property (5.70), i.e. of adding for each k, only one new vector to Π_{k-1}. Thus we have that

$$\Pi_k = \{\vec{\imath}_0, \vec{\imath}_1, \ldots, \vec{\imath}_{k-1}, \vec{\imath}_k\} = \{\vec{\imath}_0, \vec{\imath}_1, \ldots, \vec{\imath}_{k-1}\} \cup \{\vec{\imath}_k\} = \Pi_{k-1} \cup \{\vec{\imath}_k\}$$

which is equivalent with $\boldsymbol{A}^k = \boldsymbol{A}^{k-1} \cup \{\alpha_k\}$.

In Tables 6.1 and 6.2 we present $\boldsymbol{A}^k$ and $\{Z_s^k\}_{s=0}^k$ for different values of k, respectively. In addition, we refer also to Table 5.1 where one can find both $\boldsymbol{A}^7$ and Π_7.

k	α_k	$\boldsymbol{A}^k$	k	α_k	$\boldsymbol{A}^k$
0	$\frac{\pi}{2}$	$\{\frac{\pi}{2}\}$	4	$\frac{3\pi}{8}$	$\{\frac{\pi}{2}, 0, \frac{\pi}{4}, -\frac{\pi}{4}, \frac{3\pi}{8}\}$
1	0	$\{\frac{\pi}{2}, 0\}$	5	$\frac{\pi}{8}$	$\{\frac{\pi}{2}, 0, \frac{\pi}{4}, -\frac{\pi}{4}, \frac{3\pi}{8}, \frac{\pi}{8}\}$
2	$\frac{\pi}{4}$	$\{\frac{\pi}{2}, 0, \frac{\pi}{4}\}$	6	$-\frac{\pi}{8}$	$\{\frac{\pi}{2}, 0, \frac{\pi}{4}, -\frac{\pi}{4}, \frac{3\pi}{8}, \frac{\pi}{8}, -\frac{\pi}{8}\}$
3	$-\frac{\pi}{4}$	$\{\frac{\pi}{2}, 0, \frac{\pi}{4}, -\frac{\pi}{4}\}$	7	$-\frac{3\pi}{8}$	$\{\frac{\pi}{2}, 0, \frac{\pi}{4}, -\frac{\pi}{4}, \frac{3\pi}{8}, \frac{\pi}{8}, -\frac{\pi}{8}, -\frac{3\pi}{8}\}$

Table 6.1: Sets $\boldsymbol{A}^k$ associated with (6.9)

$$
\begin{array}{llll}
Z_0^0 = 1 & & & \\
Z_0^1 = z_2 e_2 & Z_1^1 = z_1 e_1 & & \\
Z_0^2 = -z_2^2 & Z_1^2 = -z_1^2 & Z_2^2 = -\frac{1}{2}(z_1 + z_2)^2 & \\
Z_0^3 = -z_2^3 e_2 & Z_1^3 = -z_1^3 e_1 & Z_2^3 = -\frac{1}{4}(z_1 + z_2)^3(e_1 + e_2) & Z_3^3 = -\frac{1}{4}(z_1 - z_2)^3(e_1 - e_2)
\end{array}
$$

Table 6.2: First pseudo-complex powers associated with (6.10)

The following lower degree examples

$$
\begin{aligned}
\mathbf{P}_2 &= \frac{1}{2}(-z_1^2 - z_2^2) = \frac{1}{2}Z_0^2 + \frac{1}{2}Z_1^2 + 0Z_2^2 \\
\mathbf{P}_3 &= \frac{1}{4}\big(- z_1^3 e_1 - \frac{1}{4}(z_1 + z_2)^3(e_1 + e_2) - \frac{1}{4}(z_1 - z_2)^3(e_1 - e_2) - z_2^3 e_2\big) \\
&= \frac{1}{4}Z_0^3 + \frac{1}{4}Z_1^3 + \frac{1}{4}Z_2^3 + \frac{1}{4}Z_3^3,
\end{aligned}
$$

extended until degree 7 by Table 6.3 suggest a unforeseen and surprisingly regular pattern. Indeed, the final general result is the following

Theorem 6.2.1 *The standard Appell polynomials $\mathbf{P}_k = \mathcal{P}_k(x_0, \underline{x}) = \mathbf{P}_k(z_1, z_2)$ can be written in terms of the special pseudo-complex powers Z_l^k of the form (6.7) with the parameter set given by (6.9) and (6.10) as following*

$$
\mathcal{P}_k(x_0, \underline{x}) = \mathbf{P}_k(z_1, z_2) = \frac{1}{2^n} \sum_{l=0}^{2^n - 1} Z_l^k(z_1, z_2), \tag{6.11}
$$

where n is the integer for which $2^n - 1 \leq k \leq 2(2^n - 1)$.

k	Coordinates of $\mathbf{P}_k$	k	Coordinates of $\mathbf{P}_k$
0	(1)	4	$\left(\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},0\right)$
1	$\left(\frac{1}{2},\frac{1}{2}\right)$	5	$\left(\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},0,0\right)$
2	$\left(\frac{1}{2},\frac{1}{2},0\right)$	6	$\left(\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},0,0,0\right)$
3	$\left(\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4}\right)$	7	$\left(\frac{1}{8},\frac{1}{8},\frac{1}{8},\frac{1}{8},\frac{1}{8},\frac{1}{8},\frac{1}{8},\frac{1}{8}\right)$

Table 6.3: Coordinates of $\mathbf{P}_k$ in the basis $\{Z_s^k\}_{s=0}^k$

Remark 6.2.2 The proof of Theorem 6.2.1 will be postponed after a number of lemmas of own interest. ♦

Remark 6.2.3 It is very important to notice that Theorem 6.2.1 was already proved in [38] applying general properties of the roots of unity, *Faà di Bruno's formula* for the derivative of a composite function, (cf. [88]) and some basic trigonometrical relationships. The decisive keystone was the aforementioned Riordan-Sofo type identity proved in this article for the first time, but through a special recurrence relation for the weighted central binomial coefficients $c_k(2)$. The aim of [38] were the different expansions in terms of different hypercomplex variables of $\mathcal{P}_k(x_0,\underline{x})$ themselves and the Riordan-Sofo type identity was detected as a necessary auxiliary tool for those expansions. Our aim here, already explained in the introductory remarks, is different and intended to obtain the Riordan-Sofo type identity as *consequence of the application of the bijective method* after proving Theorem 6.2.1 by techniques different from those used in [38]. By doing so we obtained also new summation formulae for sums of even powers of *cosine* in closed form. ♦

As previously announced, we continue now with some statements and proofs of lemmas of own interest. They are mainly based on an identity which relates a sum of even powers of *cosine* with the well known central binomial coefficients (cf. [113]). The study of closed expressions for this type of trigonometric formulae has attracted a growing interest over the past decades (cf. [13], [44] and [60]). In these references those formulae are related to probability theory, evaluation of lattice sums, among other topics. At the same time, some of them are connected with special polynomials such as Bernoulli, Euler and Chebyshev polynomials..

As stated in [13] many finite trigonometric sums remain intractable or cannot be evaluated explicitly in closed form. Fortunately there are some cases where elegant closed formulas can be found.

Lemma 6.2.4 ([113], Corollary 5) *Let $n, k \in \mathbb{N}$ such that $k < n$, then*

$$\sum_{l=1}^{\lfloor \frac{n-1}{2} \rfloor} \cos^{2k}\left(\frac{l\pi}{n}\right) = -\frac{1}{2} + \frac{n}{2^{2k+1}}\binom{2k}{k}. \tag{6.12}$$

Corollary 6.2.5 *Let $n \in \mathbb{N}_0$ and $k \in \mathbb{N}$ such that $k < 2^n$, then*

$$\frac{1}{2^n}\sum_{l=0}^{2^n-1} \cos^{2k}\left(\frac{l\pi}{2^n}\right) = c_{2k}. \tag{6.13}$$

Proof. Substituting in (6.12) n by 2^n we have that

$$\sum_{l=1}^{2^{n-1}-1} \cos^{2k}\left(\frac{l\pi}{2^n}\right) = -\frac{1}{2} + \frac{2^n}{2^{2k+1}}\binom{2k}{k}.$$

Taking into account that $\cos^{2k}\left(\frac{l\pi}{2^n}\right) = \begin{cases} 1, & l = 0 \\ 0, & l = 2^{n-1} \end{cases}$ we can decompose the sum in the left-hand side of (6.13) in the following additive way

$$\sum_{l=0}^{2^n-1} \cos^{2k}\left(\frac{l\pi}{2^n}\right) = 1 + \sum_{l=1}^{2^{n-1}-1} \cos^{2k}\left(\frac{l\pi}{2^n}\right) + \sum_{l=2^{n-1}+1}^{2^n-1} \cos^{2k}\left(\frac{l\pi}{2^n}\right). \tag{6.14}$$

We first observe that both sums on the right-hand side have the same number of summands. Moreover, it can be shown that their sum coincides. If we set in the second sum $s = 2^n - l$, i.e. $l = 2^n - s$ then

$$\sum_{s=1}^{2^{n-1}-1} \cos^{2k}\left(\frac{(2^n-s)\pi}{2^n}\right) = \sum_{s=1}^{2^{n-1}-1} \cos^{2k}\left(\pi - \frac{s\pi}{2^n}\right) = \sum_{s=1}^{2^{n-1}-1} \cos^{2k}\left(\frac{s\pi}{2^n}\right) = -\frac{1}{2} + \frac{2^n}{2^{2k+1}}\binom{2k}{k}.$$

Finally, based on (6.14) it becomes clear that

$$\sum_{l=0}^{2^n-1} \cos^{2k}\left(\frac{l\pi}{2^n}\right) = 1 + 2\left(-\frac{1}{2} + \frac{2^n}{2^{2k+1}}\binom{2k}{k}\right) = 2^n c_{2k}, \tag{6.15}$$

i.e. assertion (6.13) is proved. ∎

The following result relates the sum of even powers of *cosine* evaluated at the 2^n multiples of $\frac{\pi}{2^n}$ in the real interval $[0; \pi[$ with the sum of the same function evaluated at 2^n consecutive multiples of $\frac{\pi}{2^n}$ in $[-\frac{s\pi}{2^n}; \pi - \frac{s\pi}{2^n}[$. In fact, it can be proved that for any $s \in \mathbb{Z}$ this *shift of argument* doesn't change the value of the referred sum.

Lemma 6.2.6 *Let $n, s \in \mathbb{N}_0$ and $k \in \mathbb{N}$ such that $0 \le s \le 2^n - 1$ and $k < 2^n$. Then*

$$\sum_{l=0}^{2^n-1} \cos^{2k}\left(\frac{l\pi}{2^n}\right) = \sum_{l=0}^{2^n-1} \cos^{2k}\left(\frac{l-s}{2^n}\pi\right). \tag{6.16}$$

Proof. We first decompose the sum in the right-hand side in the following way

$$\sum_{l=0}^{2^n-1} \cos^{2k}\left(\frac{l-s}{2^n}\pi\right) = \sum_{l=0}^{2^n-1} \cos^{2k}\left(\frac{s-l}{2^n}\pi\right) = \sum_{l=0}^{s} \cos^{2k}\left(\frac{s-l}{2^n}\pi\right) + \sum_{l=s+1}^{2^n-1} \cos^{2k}\left(\frac{s-l}{2^n}\pi\right).$$

Moreover, by setting $l' = s - l$ in the first sum and $l' = 2^n + s - l$ in the second sum it follows that

$$\sum_{l=0}^{2^n-1} \cos^{2k}\left(\frac{l-s}{2^n}\pi\right) = \sum_{l'=0}^{s} \cos^{2k}\left(\frac{l'\pi}{2^n}\right) + \sum_{l'=s+1}^{2^n-1} \cos^{2k}\left(\frac{l'-2^n}{2^n}\pi\right),$$

and consequently, since $\cos^{2k}(x) = \cos^{2k}(-\pi + x)$ for any $x \in \mathbb{R}$, the last expression can be rewritten as

$$\sum_{l=0}^{2^n-1} \cos^{2k}\left(\frac{l-s}{2^n}\pi\right) = \sum_{l'=0}^{s} \cos^{2k}\left(\frac{l'\pi}{2^n}\right) + \sum_{l'=s+1}^{2^n-1} \cos^{2k}\left(\frac{l'\pi}{2^n}\right) = \sum_{l'=0}^{2^n-1} \cos^{2k}\left(\frac{l'\pi}{2^n}\right)$$

which completes the proof. ∎

Corollary 6.2.5 together with Lemma 6.2.6 leads to

Corollary 6.2.7 *The* weighted central binomial coefficients c_{2k} *can be represented by the following* cosine *power sum*

$$c_{2k} = \frac{1}{2^n} \sum_{l=0}^{2^n-1} \cos^{2k}\left(\frac{l-s}{2^n}\pi\right), \tag{6.17}$$

for arbitrary integer s and $k < 2^n$.

A slightly different formula for c_{2k} is given by

Lemma 6.2.8 *Let $n, s \in \mathbb{N}_0$ and $k \in \mathbb{N}$ such that $0 \le s \le 2^n - 1$ and $k < 2^n$. Then*

$$c_{2k} = \frac{1}{2^n} \sum_{l=0}^{2^n-1} \cos^{2k}\left(\frac{2l-s}{2^{n+1}}\pi\right). \tag{6.18}$$

Proof. We discuss separately the cases when s is even and s is odd.

If s is even it follows immediately from $0 \le s \le 2^n - 1$ that $0 \le \frac{s}{2} \le 2^{n-1} - 1 \le 2^n - 1$. Thus, by means of (6.17), we conclude for $k < 2^n$ that

$$\sum_{l=0}^{2^n-1} \cos^{2k}\left(\frac{2l-s}{2^{n+1}}\pi\right) = \sum_{l=0}^{2^n-1} \cos^{2k}\left(\frac{l-\frac{s}{2}}{2^n}\pi\right) = 2^n c_{2k},$$

i.e. the assertion.

On the other hand, if s is odd, we can decompose the sum on left-hand side of (6.18) as

$$\begin{aligned}\sum_{l=0}^{2^n-1} \cos^{2k}\left(\frac{2l-s}{2^{n+1}}\pi\right) &= \sum_{l=0}^{2^{n+1}-1} \cos^{2k}\left(\frac{l-s}{2^{n+1}}\pi\right) - \sum_{l=0}^{2^n-1} \cos^{2k}\left(\frac{2l-(s-1)}{2^{n+1}}\pi\right)\\ &= \sum_{l=0}^{2^{n+1}-1} \cos^{2k}\left(\frac{l-s}{2^{n+1}}\pi\right) - \sum_{l=0}^{2^n-1} \cos^{2k}\left(\frac{l-\frac{s-1}{2}}{2^{n}}\pi\right).\end{aligned}$$

Finally, from $k < 2^n < 2^{n+1}$ and (6.17) we conclude that

$$\sum_{l=0}^{2^n-1} \cos^{2k}\left(\frac{2l-s}{2^{n+1}}\pi\right) = 2^{n+1}c_{2k} - 2^n c_{2k} = 2^n c_{2k},$$

which completes the proof. ■

Remark 6.2.9 The previous result characterizes the sum of even powers of *cosine* evaluated at 2^n alternating multiples of $\frac{\pi}{2^{n+1}}$. During the proof it became clear that for s odd the sum (6.18) differs significantly from that of (6.17). ♦

In the following proposition one obtains the closed form of sums of odd powers of *cosine* evaluated at 2^n alternating multiples of $\frac{\pi}{2^{n+1}}$. Despite their differences with (6.18), these equalities are, as far as we know, not known. The proof is more complicated and relies on previously mentioned facts.

Proposition 6.2.10 *Let $n, s \in \mathbb{N}_0$ and $k \in \mathbb{N}$ such that $0 \leq s \leq 2^n - 1$ and $k < 2^n - 1$, then*

$$\frac{1}{2^n}\sum_{l=0}^{2^n-1} \cos^{2k+1}\left(\frac{2l-s}{2^{n+1}}\pi\right)\cos\left(\frac{l\pi}{2^n}\right) = c_{2k+1}\cos\left(\frac{s\pi}{2^{n+1}}\right) \tag{6.19}$$

$$\frac{1}{2^n}\sum_{l=0}^{2^n-1} \cos^{2k+1}\left(\frac{2l-s}{2^{n+1}}\pi\right)\sin\left(\frac{l\pi}{2^n}\right) = c_{2k+1}\sin\left(\frac{s\pi}{2^{n+1}}\right) \tag{6.20}$$

Proof. We start by rewriting $\cos\left(\frac{l\pi}{2^n}\right)$ as

$$\begin{aligned}\cos\left(\frac{l\pi}{2^n}\right) &= \cos\left(\frac{2l-s}{2^{n+1}}\pi + \frac{s}{2^{n+1}}\pi\right) &(6.21)\\ &= \cos\left(\frac{2l-s}{2^{n+1}}\pi\right)\cos\left(\frac{s}{2^{n+1}}\pi\right) - \sin\left(\frac{2l-s}{2^{n+1}}\pi\right)\sin\left(\frac{s}{2^{n+1}}\pi\right). &(6.22)\end{aligned}$$

Hence

$$\begin{aligned}\sum_{l=0}^{2^n-1} \cos^{2k+1}\left(\frac{2l-s}{2^{n+1}}\pi\right)\cos\left(\frac{l\pi}{2^n}\right) &= \cos\left(\frac{s}{2^{n+1}}\pi\right)\sum_{l=0}^{2^n-1}\cos^{2k+2}\left(\frac{2l-s}{2^{n+1}}\pi\right)\\ &\quad - \sin\left(\frac{s}{2^{n+1}}\pi\right)\sum_{l=0}^{2^n-1}\cos^{2k+1}\left(\frac{2l-s}{2^{n+1}}\pi\right)\sin\left(\frac{2l-s}{2^{n+1}}\pi\right). \qquad (6.23)\end{aligned}$$

We consider now both sums separately. As far as the first sum is concerned, we recall relation (6.18), which allows to conclude that for $k < 2^n - 1$ one has

$$\sum_{l=0}^{2^n-1} \cos^{2k+2}\left(\frac{2l-s}{2^{n+1}}\pi\right) = 2^n c_{2k+2}.$$

In addition, since $c_{2k+2} = c_{2k+1}$ (see (1.11)), it is possible to rewrite (6.23) in the following way

$$\sum_{l=0}^{2^n-1} \cos^{2k+1}\left(\frac{2l-s}{2^{n+1}}\pi\right) \cos\left(\frac{l\pi}{2^n}\right) = 2^n c_{2k+1} \cos\left(\frac{s}{2^{n+1}}\pi\right) + \sin\left(\frac{s}{2^{n+1}}\pi\right) A(k,n,s), \quad (6.24)$$

where

$$A(k,n,s) = \sum_{l=0}^{2^n-1} \cos^{2k+1}\left(\frac{2l-s}{2^{n+1}}\pi\right) \sin\left(\frac{2l-s}{2^{n+1}}\pi\right).$$

In what follows we prove that $A(k,n,s) = 0$. We start by splitting $A(k,n,s)$ as

$$A(k,n,s) = A_1(k,n,s) + A_2(k,n,s), \quad (6.25)$$

where

$$A_1(k,n,s) = \sum_{l=0}^{s} \cos^{2k+1}\left(\frac{2l-s}{2^{n+1}}\pi\right) \sin\left(\frac{2l-s}{2^{n+1}}\pi\right) \quad (6.26)$$

$$A_2(k,n,s) = \sum_{l=s+1}^{2^n-1} \cos^{2k+1}\left(\frac{2l-s}{2^{n+1}}\pi\right) \sin\left(\frac{2l-s}{2^{n+1}}\pi\right). \quad (6.27)$$

The proof of $A(k,n,s) = 0$ consists on demonstrating that $A_1(k,n,s) = A_2(k,n,s) = 0$. We first decompose $A_1 = A_1(k,n,s)$ in the following additive way

$$A_1 = \sum_{l=0}^{\lfloor \frac{s-1}{2} \rfloor} \cos^{2k+1}\left(\frac{2l-s}{2^{n+1}}\pi\right) \sin\left(\frac{2l-s}{2^{n+1}}\pi\right) + \sum_{l=\lfloor \frac{s-1}{2} \rfloor+1}^{s} \cos^{2k+1}\left(\frac{2l-s}{2^{n+1}}\pi\right) \sin\left(\frac{2l-s}{2^{n+1}}\pi\right).$$

We proceed with the following change of indexes: $l' = l+1$ in the first sum and $l' = l - \lfloor \frac{s-1}{2} \rfloor$. Based on this we can write

$$\begin{aligned} A_1 = &\sum_{l'=1}^{\lfloor \frac{s-1}{2} \rfloor+1} \cos^{2k+1}\left(\frac{2(l'-1)-s}{2^{n+1}}\pi\right) \sin\left(\frac{2(l'-1)-s}{2^{n+1}}\pi\right) \\ &+ \sum_{l'=1}^{s-\lfloor \frac{s-1}{2} \rfloor} \cos^{2k+1}\left(\frac{2l'+2\lfloor \frac{s-1}{2} \rfloor - s}{2^{n+1}}\pi\right) \sin\left(\frac{2l'-2\lfloor \frac{s-1}{2} \rfloor - s}{2^{n+1}}\pi\right). \end{aligned} \quad (6.28)$$

A detailed analysis about the upper limits of these two last sums is now required.

$$\lfloor \tfrac{s-1}{2} \rfloor + 1 = \begin{cases} \frac{s}{2} & \text{if } s \text{ is even} \\ \frac{s+1}{2} & \text{if } s \text{ is odd} \end{cases} \tag{6.29}$$

$$s - \lfloor \tfrac{s-1}{2} \rfloor = \begin{cases} \frac{s}{2} + 1 & \text{if } s \text{ is even} \\ \frac{s+1}{2} & \text{if } s \text{ is odd} \end{cases} \tag{6.30}$$

Although the upper limits differ for the case when s is even, we observe that the last sum of (6.28) vanishes for $l = \frac{s}{2} + 1$. Based on this its upper limit can be changed to $s - \lfloor \frac{s-1}{2} \rfloor$. In addition, by considering in the last sum of (6.28) the change of index $l = s - \lfloor \frac{s-1}{2} \rfloor - l' + 1$ we obtain from $1 \leq l' \leq s - \lfloor \frac{s-1}{2} \rfloor$ also that $1 \leq l \leq s - \lfloor \frac{s-1}{2} \rfloor$ and therefore

$$\begin{aligned} A_1 = & \sum_{l=1}^{s-\lfloor \frac{s-1}{2} \rfloor} \cos^{2k+1}\left(\frac{2l-s-2}{2^{n+1}}\pi\right) \sin\left(\frac{2l-s-2}{2^{n+1}}\pi\right) \\ & + \sum_{l=1}^{s-\lfloor \frac{s-1}{2} \rfloor} \cos^{2k+1}\left(-\frac{2l-s-2}{2^{n+1}}\pi\right) \sin\left(-\frac{2l-s-2}{2^{n+1}}\pi\right). \end{aligned} \tag{6.31}$$

Finally, since for $x \in \mathbb{R}$ we have that $\cos^{2k+1}(-x)\sin(-x) = -\cos^{2k+1}(x)\sin(x)$, then we conclude that $A_1 = 0$.

We will now analyze (6.27), i.e. $A_2 = A_2(k, n, s)$ and we start by splitting the sum as

$$\begin{aligned} A_2 = & \sum_{l=s+1}^{2^{n-1}+\lfloor \frac{s-1}{2} \rfloor} \cos^{2k+1}\left(\frac{2l-s}{2^{n+1}}\pi\right) \sin\left(\frac{2l-s}{2^{n+1}}\pi\right) \\ & + \sum_{l=2^{n-1}+\lfloor \frac{s-1}{2} \rfloor+1}^{2^n-1} \cos^{2k+1}\left(\frac{2l-s}{2^{n+1}}\pi\right) \sin\left(\frac{2l-s}{2^{n+1}}\pi\right). \end{aligned} \tag{6.32}$$

As far as the last sum is concerned we will change its lower limit based on the following reasoning. By comparing

$$\lfloor \tfrac{s}{2} \rfloor + 1 = \begin{cases} \frac{s}{2} + 1 & \text{if } s \text{ is even} \\ \frac{s+1}{2} & \text{if } s \text{ is odd} \end{cases}.$$

with (6.29), it is clear that $\lfloor \frac{s}{2} \rfloor + 1$ and $\lfloor \frac{s-1}{2} \rfloor + 1$ differ only for the case when s is even. Even though it is easy to see that in this case, for $l = 2^{n-1} + \frac{s}{2}$ the corresponding term is

$$\cos^{2k+1}\left(\frac{2^n}{2^{n+1}}\pi\right) \sin\left(\frac{2^n}{2^{n+1}}\pi\right) = 0,$$

which justifies the change of the lower limit of the last sum of (6.32) to $\lfloor \frac{s}{2} \rfloor + 1$. Based on this, and by considering the change of index $l' = l - 2^{n-1} + \lfloor \frac{s-1}{2} \rfloor + 1$, the second sum of expression (6.32) takes the form

$$\sum_{l=2^{n-1}+\lfloor \frac{s}{2} \rfloor+1}^{2^n-1} \cos^{2k+1}\Big(\frac{2l-s}{2^{n+1}}\pi\Big)\sin\Big(\frac{2l-s}{2^{n+1}}\pi\Big)$$
$$= \sum_{l'=\lfloor \frac{s}{2} \rfloor+\lfloor \frac{s-1}{2} \rfloor+2}^{2^{n-1}+\lfloor \frac{s-1}{2} \rfloor} \cos^{2k+1}\Big(\frac{2l'+2^n-2\lfloor \frac{s-1}{2} \rfloor-s-2}{2^{n+1}}\pi\Big)\sin\Big(\frac{2l'+2^n-2\lfloor \frac{s-1}{2} \rfloor-s-2}{2^{n+1}}\pi\Big)$$

Moreover, since $\lfloor \frac{s}{2} \rfloor + \lfloor \frac{s-1}{2} \rfloor + 2 = s+1$, we may rewrite last expression by reversing the order of the terms in the summation, that is by considering $l' = 2^{n-1} + \lfloor \frac{s-1}{2} \rfloor + s + 1 - l$, in such a way that expression (6.32) takes the form

$$A_2 = \sum_{l=s+1}^{2^{n-1}+\lfloor \frac{s-1}{2} \rfloor} \cos^{2k+1}\Big(\frac{2l-s}{2^{n+1}}\pi\Big)\sin\Big(\frac{2l-s}{2^{n+1}}\pi\Big)$$
$$+ \sum_{l=s+1}^{2^{n-1}+\lfloor \frac{s-1}{2} \rfloor} \cos^{2k+1}\Big(\frac{2^{n+1}+2s-2l}{2^{n+1}}\pi\Big)\sin\Big(\frac{2^{n+1}+2s-2l}{2^{n+1}}\pi\Big).$$

With the knowledge that $\cos^{2k+1}(\pi - x)\sin(\pi - x) = -\cos^{2k+1}(x)\sin(x)$, for $x \in \mathbb{R}$, it follows at once that $A_2 = 0$. Thus, according to (6.24) we have demonstrated that $A(k, n, s) = 0$ and therefore assertion (6.19) is proved.

With regard to (6.20), its proof is completely analogous to the proof of (6.19). ■

Remark 6.2.11 It is possible to show that relation (6.16) is true for any $s \in \mathbb{Z}$. We omit the proof which is analogous to the proof of Lemma 6.2.6. ♦

We are now able to prove Theorem 6.2.1 which will allow us to reach the goal of this chapter and to obtain the announced Riordan-Sofo type binomial identity (6.4)(cf. Theorem 2 in [38]) by a bijection proof.

PROOF of THEOREM 6.2.1

Denote $\widetilde{Z}_k = \frac{1}{2^n}\sum_{l=0}^{2^n-1} Z_l^k(z_1, z_2)$. According to the representation of the PCP in terms of x_0, x_1 and x_2 we have that

$$\widetilde{Z}_k(x_0, x_1, x_2) = \frac{1}{2^n}\sum_{l=0}^{2^n-1}\Big(x_0 + \big(x_1\cos\frac{l\pi}{2^n} + x_2\sin\frac{l\pi}{2^n}\big)\big(\cos\frac{l}{2^n}\pi e_1 + \sin\frac{l}{2^n}\pi e_2\big)\Big)^k \qquad (6.33)$$

Both polynomials (6.11) and (6.33) are homogeneous paravector valued polynomials of degree k with real constant coefficients and the common coefficient of x_0^k equal to 1. According to this, the proof consists on showing that they coincide at k distinct points on the unit sphere in the hyperplane $x_0 = 0$ of the form

$$x_s = \underline{x}_s = \cos\frac{s\pi}{2^{n+1}}e_1 + \sin\frac{s\pi}{2^{n+1}}e_2, \tag{6.34}$$

where $s = 0, 1, \ldots, k-1$, with $2^n - 1 \leq k \leq 2(2^n - 1)$. We distinguish two different cases depending on the parity of k.

***k* is even:**

The evaluation of (6.11) and (6.33) at (6.34) gives

$$\mathcal{P}_k(0, \cos\frac{s\pi}{2^{n+1}}e_1 + \sin\frac{s\pi}{2^{n+1}}e_2) = (-1)^{\frac{k}{2}}c_k$$

and

$$\begin{aligned}\widetilde{Z}_k(0, \cos\frac{s\pi}{2^{n+1}}, \sin\frac{s\pi}{2^{n+1}}) &= \frac{1}{2^n}\sum_{l=0}^{2^n-1}\Big(\cos\frac{s\pi}{2^{n+1}}\cos\frac{l\pi}{2^n} + \sin\frac{s\pi}{2^{n+1}}\sin\frac{l\pi}{2^n}\Big)^k \\ &= \frac{(-1)^{\frac{k}{2}}}{2^n}\sum_{l=0}^{2^n-1}\cos^k\Big(\frac{s-2l}{2^{n+1}}\pi\Big).\end{aligned}$$

By recalling (6.18), since $k < 2^{n+1}$ and therefore $\frac{k}{2} < 2^n$, we conclude that both $\mathcal{P}_k$ and $\widetilde{Z}_k$ coincide at the referred points.

***k* is odd:**

In this case, since $\underline{x}^k = (-1)^{\frac{k-1}{2}}\underline{x}$, by using a similar argument as the one used for the previous case, allows us to obtain

$$\mathcal{P}_k\Big(0, \cos\frac{s\pi}{2^{n+1}}e_1 + \sin\frac{s\pi}{2^{n+1}}e_2) = c_k\underline{x}^k = (-1)^{\frac{k-1}{2}}c_k(\cos\frac{s\pi}{2^{n+1}}e_1 + \sin\frac{s\pi}{2^{n+1}}e_2) \tag{6.35}$$

and

$$\widetilde{Z}_k(0, \cos\frac{s\pi}{2^{n+1}}, \sin\frac{s\pi}{2^{n+1}}) = \frac{(-1)^{\frac{k-1}{2}}}{2^n}\sum_{l=0}^{2^n-1}\cos^k\Big(\frac{s-2l}{2^{n+1}}\pi\Big)(\cos\frac{l\pi}{2^n}e_1 + \sin\frac{l\pi}{2^n}e_2). \tag{6.36}$$

Comparing now (6.35) and (6.36) by using the relations (6.19) and (6.20), we verify immediately that both expressions are equivalent and Theorem 6.2.1 is proved. ■

Applications of Theorem 6.2.1

We can now derive new identities where the coefficients c_k play a central role.

The first step is

Proposition 6.2.12 *Let k be an even number with $2^n - 1 \leq k \leq 2(2^n-1)$, $s = 0, 1, \ldots, \lfloor \frac{k}{2} \rfloor$ and $\beta_l = \frac{l\pi}{2^n}$ then, for any $t \in \mathbb{R}$:*

$$c_{2s}(1+t^2)^s = \frac{1}{2^n} \sum_{l=0}^{2^n-1} \Big(\cos\beta_l + t \sin\beta_l \Big)^{2s} \tag{6.37}$$

$$c_{2s+1}(1+t^2)^s = \frac{1}{2^n} \sum_{l=0}^{2^n-1} \Big(\cos\beta_l + t\beta_l \Big)^{2s+1} \cos\beta_l \tag{6.38}$$

$$c_{2s+1}(1+t^2)^s = \frac{1}{2^n} \sum_{l=0}^{2^n-1} \Big(\cos\beta_l + t\beta_l \Big)^{2s+1} t \sin\beta_l. \tag{6.39}$$

Proof. Consider the polynomials (6.5) and (6.33), both evaluated at

$$x = 1 + te_1 + t^2 e_2. \tag{6.40}$$

We prove (6.37) by comparing the real parts of $\mathcal{P}_k$ and $\widetilde{Z}_k$ evaluated at (6.40). The formulae (6.38) and (6.39) follow analogously by comparing the coefficients of e_1 and e_2 of both polynomials, respectively.

In this case we have

$$\mathcal{P}_k(1, te_1 + t^2 e_2) = \sum_{s=0}^{k} \binom{k}{s} c_s (te_1 + t^2 e_2)^s,$$

and consequently,

$$\begin{aligned} \mathcal{P}_k(1, te_1 + t^2 e_2) = & \sum_{s=0}^{\lfloor \frac{k}{2} \rfloor} \binom{k}{2s} (-1)^s t^{2s} c_{2s} (1+t^2)^s \\ & + \sum_{s=0}^{\lfloor \frac{k-1}{2} \rfloor} \binom{k}{2s+1} (-1)^s t^{2s+1} c_{2s+1} (1+t^2)^s (e_1 + te_2). \end{aligned} \tag{6.41}$$

Regarding the PCP Z_l^k

$$\begin{aligned} Z_l^k(1, t, t^2) &= \sum_{s=0}^{k} \binom{k}{s} [t(\cos\beta_l + t\sin\beta_l)(\cos\beta_l e_1 + \sin\beta_l e_2)]^s \\ &= \sum_{s=0}^{\lfloor \frac{k}{2} \rfloor} \binom{k}{2s} (-1)^s (\cos\beta_l + t\sin\beta_l)^{2s} t^{2s} \\ &+ \sum_{s=0}^{\lfloor \frac{k-1}{2} \rfloor} \binom{k}{2s+1} (-1)^s (\cos\beta_l + t\sin\beta_l)^{2s+1} t^{2s+1} (\cos\beta_l e_1 + t\sin\beta_l e_2), \end{aligned}$$

which, according to (6.33), gives

$$\widetilde{Z}_k(1,t,t^2) = \sum_{s=0}^{\lfloor \frac{k}{2} \rfloor} \binom{k}{2s} (-1)^s t^{2s} \frac{1}{2^n} \sum_{l=0}^{2^n-1} (\cos\beta_l + t\sin\beta_l)^{2s}$$
$$+ \sum_{s=0}^{\lfloor \frac{k-1}{2} \rfloor} \binom{k}{2s+1} (-1)^s t^{2s+1} \frac{1}{2^n} \sum_{l=0}^{2^n-1} (\cos\beta_l + t\sin\beta_l)^{2s+1} (\cos\beta_l e_1 + t\sin\beta_l e_2). \tag{6.42}$$

By comparing (6.41) with (6.42), formulae (6.37)-(6.39) follow at once. ■

Remark 6.2.13 Observe that relation (6.13) corresponds to the particular case of (6.37) evaluated at $t = 0$. ♦

In the following corollary one can find closed expressions of finite sums of products of even (resp. odd) powers of trigonometric functions.

Using the binomial expansion of both sides of (6.37) one obtains

$$\sum_{u=0}^{s} \binom{s}{u} c_{2s} t^{2u} = \sum_{u=0}^{2s} \left[\binom{2s}{u} \frac{1}{2^n} \sum_{l=0}^{2^n-1} (\cos\beta_l)^{2s-u} (\sin\beta_l)^u \right] t^u.$$

Notice that the left-hand side contains only even powers of t. Hence, by splitting the right-hand side in terms of even and odd powers of the variable t and comparing the corresponding powers with the left-hand side, the next corollary follows at once.

Corollary 6.2.14 *Let $k, n \in \mathbb{N}_0$ such that $s = 0, 1, \ldots, \lfloor \frac{k}{2} \rfloor$ with $2^n - 1 \leq k \leq 2(2^n - 1)$ and $\beta_l = \frac{l\pi}{2^n}$. Then*

$$\frac{1}{2^n} \sum_{l=0}^{2^n-1} \big(\cos\beta_l\big)^{2s-2u} \big(\sin\beta_l\big)^{2u} = c_{2s} \binom{s}{u} \binom{2s}{2u}^{-1} \tag{6.43}$$

$$\frac{1}{2^n} \sum_{l=0}^{2^n-1} \big(\cos\beta_l\big)^{2s-2u-1} \big(\sin\beta_l\big)^{2u+1} = 0. \tag{6.44}$$

The next proposition was already used in [38] and will serve as an auxiliary result for the proof of the Riordan-Sofo type binomial identity.

Proposition 6.2.15 *If N and p are integers such that $N \geq 0$ and $p \neq kN$, for odd k, then the following equalities are true:*

$$\sum_{l=1}^{2N} (-1)^l \cos(lp\tfrac{\pi}{N}) = \sum_{l=1}^{2N} (-1)^l \sin(lp\tfrac{\pi}{N}) = 0. \tag{6.45}$$

Proof. Let $\omega = \cos(\frac{\pi}{N}) + i\sin(\frac{\pi}{N})$ be a $2N$-*th* primitive root of unity. Since $\omega \neq 1$ we have

$$\sum_{l=1}^{2N} \omega^l = 0. \tag{6.46}$$

We note that the assumption $p \neq kN$, for odd k, is equivalent to assume that $\omega^p \neq -1$. In this way, replacing ω by $-\omega^p$ in formula (6.46), we obtain

$$\sum_{l=1}^{2N} (-\omega^p)^l = 0,$$

or equivalently,

$$\sum_{l=1}^{2N} (-1)^l \cos(lp\tfrac{\pi}{N}) + i\sum_{l=1}^{2N} (-1)^l \sin(lp\tfrac{\pi}{N}) = 0.$$

■

We recall here a well known expression for even powers of the *cosine* function. Let N be a positive integer, then

$$(\cos\theta)^{2N} = \frac{1}{2^{2N}}\binom{2N}{N} + \frac{1}{2^{2N-1}}\sum_{t=0}^{N-1}\binom{2N}{t}\cos(2(N-t)\theta),$$

which can be written in terms of the coefficients c_k as

$$(\cos\theta)^{2N} = c_{2N} + \frac{1}{2^{2N-1}}\sum_{t=0}^{N-1}\binom{2N}{t}\cos(2(N-t)\theta). \tag{6.47}$$

Finally, we derive the binomial identity that generalizes the Riordan-Sofo identity (6.3).

Proposition 6.2.16 (Riordan-Sofo type identity) *If $s \in \mathbb{N}$ and $u = 0, 1, \ldots, s$, then*

$$\sum_{t=0}^{u} (-1)^{u-t}\binom{u}{t} c_{2(s-t)} = \binom{s}{u}\binom{2s}{2u}^{-1} c_{2s}. \tag{6.48}$$

Proof. The proof relies on some previously from Theorem 6.2.1 derived formulae. We start by denoting the left-hand side of (6.43) by

$$F_s(u) = \frac{1}{2^n}\sum_{l=0}^{2^n-1}\big(\cos\beta_l\big)^{2s-2u}\big(\sin\beta_l\big)^{2u},$$

where u and s vary according to Corollary 6.2.14. This expression can be rewritten as

$$F_s(u) = \frac{1}{2^n}\sum_{l=1}^{2^n}\big(\cos\beta_l\big)^{2s-2u}\big(1-\cos^2\beta_l\big)^u = \frac{1}{2^n}\sum_{t=0}^{u}(-1)^t\binom{u}{t}\sum_{l=1}^{2^n}\big(\cos\beta_l\big)^{2s-2(u-t)}.$$

By applying to this last equality relation (6.47) it follows that

$$F_s(u) = \frac{1}{2^n} \sum_{t=0}^{u} (-1)^t \binom{u}{t} 2^n c_{2(s-u+t)} + A(s,u), \tag{6.49}$$

where

$$A(s,u) = \frac{1}{2^n} \sum_{t=0}^{u} \binom{u}{t} \frac{(-1)^t}{2^{2(s-u+t)-1}} \sum_{r=0}^{s-u+t-1} \binom{2(s-u+t)}{r} \sum_{l=1}^{2^n} \cos\big(2(s-u+t-r)\beta_l\big).$$

Since $2^n - 1 \leq 2s \leq 2(2^n - 1)$ we can write $2s = 2^n + 2s'$, for $0 \leq s' \leq\ 2^{n-1} - 1$ and conclude that

$$\sum_{l=1}^{2^n} \cos\big(2(s-u+t-r)\tfrac{l\pi}{2^n}\big) = \sum_{l=1}^{2^n} (-1)^l \cos\big((s'-u+t-r)\tfrac{l\pi}{2^{n-1}}\big).$$

Now, in order to be able to apply Proposition 6.2.15, we need to ensure that $s' - u + t - r$ is not an odd multiple of 2^{n-1}. Based on the fact that $0 \leq r \leq s - u + t - 1$ and $t - u \leq 0$, we are led to the relation

$$s' - s + 1 = -2^{n-1} + 1 \leq s' - u + t - r \leq s' \leq 2^{n-1} - 1$$

and consequently, by the use of Proposition 6.2.15, the aforementioned sum vanishes and therefore also the sum $A(s,u)$ in (6.49). Hence, according to (6.48), the final form of (6.49) is given by

$$F_s(u) = \sum_{t=0}^{u} (-1)^t \binom{u}{t} c_{2(s-u+t)} = \sum_{t=0}^{u} (-1)^{u-t} \binom{u}{t} c_{2(s-t)} = \binom{s}{u} \binom{2s}{2u}^{-1} c_{2s}, \tag{6.50}$$

and the Riordan-Sofo type identity 6.2.16 is proved. ■

6.3 Applications

The aim of this section is to collect several representations of the sequence formed by weighted central binomial coefficients which, in some of the cases, are presented here for the first time.

We consider now the following representation of the well known Legendre polynomials (cf. [133], formula 3.137)

$$\mathcal{L}_k(x) = x^k \sum_{s=0}^{\left[\frac{k}{2}\right]} \binom{k}{2s} \binom{2s}{s} \Big(\frac{x^2-1}{4x^2}\Big)^s, \quad x \in \mathbb{R}. \tag{6.51}$$

Proposition 6.3.1 *Let $k \in \mathbb{N}_0$ and $\mathcal{P}_k(x_0, \underline{x})$ be the SAP* (6.5). *Then the following relation holds*

$$Re[\mathcal{P}_k(x_0, \underline{x})] = |x|^k \mathcal{L}_k\Big(\frac{x_0}{|x|}\Big). \tag{6.52}$$

Proof.

Formula (6.51) can be written in terms of the coefficients c_k as

$$\mathcal{L}_k(x) = x^k \sum_{s=0}^{\left[\frac{k}{2}\right]} c_{2s} \binom{k}{2s} \Big(\frac{x^2-1}{x^2}\Big)^s, \tag{6.53}$$

and therefore,

$$\mathcal{L}_k\left(\frac{x_0}{|x|}\right) = \left(\frac{x_0}{|x|}\right)^k \sum_{s=0}^{\left[\frac{k}{2}\right]} c_{2s} \binom{k}{2s} \left(\frac{x_0^2 - |x|^2}{x_0^2}\right)^s = \frac{1}{|x|^k} \sum_{s=0}^{\left[\frac{k}{2}\right]} c_{2s} \binom{k}{2s} x_0^{k-2s} \underline{x}^{2s}.$$

Thus, by recalling the $n = 2$ form of (2.12), the assertion follows now at once. ■

For the special case of $x_0 = 0$ and $|x| = |\underline{x}| = 1$, and whenever k is even, relation (6.53) provides a *new* form of representing the coefficients c_k. In concrete, by observing that for odd k one has $\mathcal{L}_k(0) = 0$, the next result follows immediately.

Corollary 6.3.2 *Let $k \in \mathbb{N}_0$ then,*

$$\mathcal{L}_k(0) = \begin{cases} (-1)^{\frac{k}{2}} c_k & \text{if } k \text{ is even} \\ 0 & \text{if } k \text{ is odd} \end{cases}. \tag{6.54}$$

In the following result we express the SAP in terms of the well known Bessel functions of the first kind J_r of order r. To that end, we consider here the expression of the holomorphic exponencial Exp_n −function defined in $\mathcal{A}_n$ in [27] in terms of the SAP. For $n = 2$ it is given by

$$\mathrm{Exp}_2(x) = \mathrm{Exp}_2(x_0 + \underline{x}) = \sum_{k=0}^{\infty} \frac{\mathcal{P}_k(x)}{k!} = e^{x_0}\Big(J_0(|\underline{x}|) + \omega(\underline{x}) J_1(|\underline{x}|)\Big). \tag{6.55}$$

Proposition 6.3.3 *Let $k \in \mathbb{N}_0$ and $\omega(\underline{x}) = \frac{\underline{x}}{|\underline{x}|}$, $\underline{x} \neq 0$, then*

$$\mathcal{P}_k(x_0, \underline{x}) = \sum_{s=0}^{k} \binom{k}{s} x_0^{k-s} |\underline{x}|^s \Big(J_0^{(s)}(0) + \omega(\underline{x}) J_1^{(s)}(0)\Big), \tag{6.56}$$

where $J_0^{(s)}(0)$ and $J_1^{(s)}(0)$ denote the $s-$th derivative of J_0 and J_1 evaluated at 0, respectively.

Proof. Using the fact that $\mathcal{P}_k$ is a homogeneous polynomial of degree k, from (6.55) it follows that

$$\mathrm{Exp}_2(xt) = \sum_{k=0}^{\infty} \frac{\mathcal{P}_k(x)t^k}{k!}, \quad t \in \mathbb{R},$$

and therefore, based on (6.55) one obtains

$$\mathcal{P}_k(x) = \frac{\partial^k}{\partial t^k}\left[e^{x_0 t}\big(J_0(|\underline{x}|t) + \omega(\underline{x})J_1(|\underline{x}|t)\big)\right]\Big|_{t=0}.$$

Finally, by applying the Leibniz's rule for the $k-$th derivative of a product we obtain explicitly (6.56). ∎

Corollary 6.3.4 *For $k \in \mathbb{N}_0$ the following holds*

$$J_0^{(k)}(0) = \begin{cases} 0 & \text{if } k \text{ is odd} \\ (-1)^{\frac{k}{2}} c_k & \text{if } k \text{ is even} \end{cases} \tag{6.57}$$

$$J_1^{(k)}(0) = \begin{cases} (-1)^{\frac{k-1}{2}} c_k & \text{if } k \text{ is odd} \\ 0 & \text{if } k \text{ is even} \end{cases}. \tag{6.58}$$

Proof. Consider in expression (6.56) the case of $x_0 = 0$ and $x = \underline{x}$ with $|\underline{x}| = 1$. ∎

Comments to Chapter 6

In what follows we give a list of several representations of the coefficients c_k.

1. Central binomial coefficients (see (2.18))

$$c_k = \frac{1}{2^k}\binom{k}{\lfloor \frac{k}{2} \rfloor}. \tag{6.59}$$

2. Double factorials (cf. with (2.10) for $n = 2$)

$$c_k = \begin{cases} \frac{k!!}{(k+1)!!} & \text{if } k \text{ is odd} \\ c_{k-1} & \text{if } k \text{ is even} \end{cases}. \tag{6.60}$$

3. Coefficients $T_s^k = T_s^k(2)$ (cf. with (2.7) for $n = 2$)

$$c_k = \sum_{s=0}^{k}(-1)^s T_s^k = \sum_{s=0}^{k} \frac{(-1)^s}{k+1} \frac{\left(\frac{3}{2}\right)_{(k-s)}\left(\frac{1}{2}\right)_{(s)}}{(k-s)!s!}. \tag{6.61}$$

4. Reciprocal of a finite sum of hypercomplex numbers (see (2.9) for $n = 2$)

$$c_k = \left[\sum_{s=0}^{k} \binom{k}{s} (e_1^{k-s} \times e_2^{s})^2 \right]^{-1} = \left[\sum_{s=0}^{k} \binom{k}{s}^{-1} \mathcal{E}^2_{(k,s)} \right]^{-1}, \tag{6.62}$$

where $\mathcal{E}_{(k,s)}$ are the hypercomplex numbers (3.9).

5. Reciprocal of a finite sum (cf. [74], page 24, Vol. 4)

$$c_{2k} = \left[\sum_{s=0}^{k} \binom{k}{s}^2 \binom{2k}{2s}^{-1} \right]^{-1}. \tag{6.63}$$

6. Finite sum of previous values (cf. [74], page 17, Vol. 3)

$$c_{2k} = \frac{1}{2k} \sum_{s=0}^{k-1} c_{2s}. \tag{6.64}$$

7. Reed Dawson identity (see (6.1))

$$c_{2k} = \sum_{s=0}^{2k} \frac{(-1)^s}{2^s} \binom{2s}{s} \binom{2k}{s} = \sum_{s=0}^{2k} (-2)^s \binom{2k}{s} c_{2s}. \tag{6.65}$$

8. Finite alternating sum (cf. [74], page 21, Vol. 6)

$$c_{2k} = \sum_{s=0}^{k} (-1)^s \binom{k}{s} c_{2s}. \tag{6.66}$$

9. Cauchy convolution formula (cf. [74], page 31, Vol. 1)

$$c_{2k} = \sum_{s=0}^{k} \binom{k}{s}^2. \tag{6.67}$$

10. Infinite series (cf. [74], page 5, Vol. 2)

$$c_{2k} = \frac{1}{\sqrt{2}} \sum_{s=0}^{+\infty} \frac{1}{2^{3s}} \binom{2s}{s} \binom{s}{k}. \tag{6.68}$$

11. Riordan-Sofo type identity (6.48) where, for $u = 0, 1, \ldots, \lfloor \frac{k}{2} \rfloor$ we have

$$c_{2k} = \binom{2k}{2u} \binom{k}{u}^{-1} \sum_{l=0}^{u} (-1)^{u-l} \binom{l}{u} c_{2(k-l)}. \tag{6.69}$$

12. Finite sum of even powers of trigonometric functions (see (6.13), (6.17) and (6.18))

$$c_{2k} = \frac{1}{2^n} \sum_{l=0}^{2^n-1} \cos^{2k}\left(\frac{l\pi}{2^n}\right) = \frac{1}{2^n} \sum_{l=0}^{2^n-1} \cos^{2k}\left(\frac{s-l}{2^n}\pi\right) = \frac{1}{2^n} \sum_{l=0}^{2^n-1} \cos^{2k}\left(\frac{s-2l}{2^{n+1}}\pi\right). \quad (6.70)$$

13. Integral representation (obtained through the famous Wallis formula)

$$c_k = \begin{cases} \frac{1}{k+1}\big(I(k)\big)^{-1}, & \text{if } k \text{ is } \textit{odd} \\ \frac{2}{\pi} I(k), & \text{if } k \text{ is } \textit{even} \end{cases}, \quad (6.71)$$

where

$$I(k) = \int_0^{\frac{\pi}{2}} \cos^k(x) dx.$$

14. Legendre polynomials (see (6.54))

$$c_{2k} = (-1)^k \mathcal{L}_{2k}(0). \quad (6.72)$$

15. Bessel functions of the first kind (see (6.57) and (6.58))

$$c_k = \begin{cases} (-1)^{\frac{k-1}{2}} J_1^{(k)}(0) & \text{if } k \text{ is } \textit{odd} \\ (-1)^{\frac{k}{2}} J_0^{(k)}(0) & \text{if } k \text{ is } \textit{even} \end{cases} \quad (6.73)$$

where $J_0^{(k)}(0)$ and $J_1^{(k)}(0)$ denote the $k-$th derivative of the Bessel functions of the first kind evaluated at 0, respectively.

Conclusion and Outlook

The aim of this thesis was the study of different aspects of Clifford algebra-valued generalized Appell polynomials, particularly some numerical and combinatorial applications.

The fact that the set of Clifford holomorphic functions is not closed neither under multiplication nor composition implies fundamental problems for the extension of complex function theory to hypercomplex function theory (HFT). This is mainly caused by the use of the noncommutative Clifford Algebra $\mathcal{C}\ell_{0,n}$. As an example of such a fundamental problem we studied in this thesis the construction of Clifford holomorphic polynomials by linear combinations of monomials built by powers of one variable. It can only be done in the usual way of classical complex function theory, if the monomials are thought as powers of some underlying totally regular variable (TRV). This led us to the systematical study of sets of pseudo-complex powers (PCP) and their applications in Chapters 4-6. Another way out of this situation could be the consideration of polynomials as linear combinations of some special sequences of polynomials of different homogeneous degrees that behave almost like the usual monomials. This idea led in the recent past several authors to the consideration of Clifford algebra-valued generalized Appell polynomials which include as special cases also pseudo-complex powers. In Chapter 2 and 3 we contributed to the study of the sequence of Standard Appell polynomials (SAP) by constructing generalized Joukowski transformations in $\mathbb{R}^3$ and investigating properties of the related Pascal $n-$simplex with hypercomplex entries. Chapter 4 includes new results about special sets of PCP and SAP showing their role as realizations of general paravector valued Appell polynomials of paravector variables in the cases $n=2$ and $n=3$.

Concerning main properties of sets of PCP from the numerical point of view, we analyzed in Chapter 5 the computational effort needed to construct a basis of PCP for the space $\mathcal{M}_k(\mathbb{H},\mathbb{H})$ of $\mathbb{H}-$linear homogeneous holomorphic $\mathbb{H}-$valued polynomials of degree k and compared it with the use of generalized powers (GP). The isomorphism of the PCP with the complex powers $z^k,\ k=0,1,\ldots$ shows the numerical efficiency of this type of generalized

Appell polynomials. Another aspect was their application for inverting all the Vandermonde matrices of order less or equal than k. Based on a recursive procedure we proposed a new algorithm which proved to be competitive compared with the Eisinberg and Fedele algorithm.

In Chapter 6 we concentrated on some combinatorial applications of generalized Appell polynomials. The comparison of two Appell polynomials with values in $\mathbb{R}^3$ led to new summation formulas of trigonometrical functions inherent in the parameterized form of PCP as well as a binomial identity involving the central binomial coefficients (generalized Riordan-Sofo identity) as results of bijective representations.

In our opinion, the results presented here give new insights into the effectiveness and usefulness of HFT methods, particularly with respect to their numerical and combinatorial applications. They could be used and further developed in different directions.

We refer, for instance, to the problem of approximating a $\mathcal{A}_2$-valued holomorphic function by means of PCP. It seems natural to think that the choice of different parameter sets will lead to different results and different algorithms that could be submitted to an adequate optimization problem.

Another aspect would be to benefit from the numerical efficiency of computation with PCP by extending results about a generalized Kantorovich method for quasiconformal transformations in [42].

Finally, another idea would be the consideration of PCP on the Chebychev nodes (5.59). In such case the associated PCP reveal a simple structure as direct consequence of the symmetry of these nodes. Therefore, it would also be natural to use PCP associated to this special type of nodes not only in approximation problems but also for obtaining new combinatorial identities.

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